# Multi-stage Stern-Gerlach experiment modeled (with additional appendices) 

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#### Abstract

In the classic multi-stage Stern-Gerlach experiment conducted by Frisch and Segrè, the Majorana (Landau-Zener) and Rabi formulae diverge afar from the experimental observation while the physical mechanism for electron-spin collapse remains unidentified. Here, introducing the physical co-quantum concept provides a plausible physical mechanism and predicts the experimental observation in absolute units without fitting (i.e., no parameters adjusted) with a $p$ value less than one per million, which is the probability that the co-quantum theory happens to match the experimental observation purely by chance. Further, the co-quantum concept is corroborated by statistically reproducing exactly the wave function, density operator, and uncertainty relation for electron spin in Stern-Gerlach experiments.


Keywords
Stern-Gerlach experiment; electron spin; Majorana formula; Landau-Zener formula; co-quantum dynamics; nonadiabatic transition; entanglement

## 1. Introduction

Performed three years before the successful development of quantum mechanics, the 1922 SternGerlach experiment on silver atoms [1] quickly proved fundamental to quantum physics [2, 3]. The benchmark experiment led to the quantization of all angular momenta, discovery of electron spin, study of the measurement problem and superposition, direct investigation of the ground-state properties of atoms without electronic excitation, and selection of fully spin-polarized atoms [2]. Within a few weeks, Einstein and Ehrenfest concluded that spin collapse cannot be interpreted by radiation, which would take 100 years [4]. Recently, Wennerström and Westlund numerically simulated that relaxation of $1 \mu \mathrm{~s}$ qualitatively reproduced the double branched collapse pattern [5], and Norsen interpreted spin collapse using the de Broglie-Bohm pilot-wave theory [6]. The significance of the Stern-Gerlach experiment and relevant works are detailed in a 2016 inspiring review [2], concluding that "The physical mechanism responsible for the alignment of the silver atoms remained and remains a mystery" and quoting Feynman, "... instead of trying to give you a theoretical explanation, we will just say that you are stuck with the result of this experiment ..." [7].

Immediately, Heisenberg and Einstein proposed multi-stage Stern-Gerlach experiments to explore deeper mysteries of directional quantization [2]. Ten years later, Phipps and Stern reported the first effort [8], which was unfortunately discontinued owing to Phipps' involuntary return to the US [2]. A year later, Frisch and Segrè modified the same apparatus by adopting Einstein's suggestion on the use of a single wire instead of three electromagnets to rotate spin; they also improved magnetic shielding, slit filtering, and signal detection [2]. Despite the use of three layers of magnetic shielding for the middle stage (i.e., the inner rotation chamber), the remnant or residual fringe magnetic field was still $0.42 \times 10^{-4} \mathrm{~T}$ (or 0.42 G ). Rather than fight the fringe magnetic field further, they took advantage of it. The magnetic field from the wire in the middle stage cancels the remnant field to produce a magnetic null point, around which the field is approximated as a magnetic quadrupole; consequently, they successfully observed nonadiabatic spin flip [9]. Note that only the magnetic field near a null point is effective for nonadiabatic spin flip; thus, the field far from a null point does not significantly affect transition, and its detailed distribution is of little import. Frisch and Segrè varied the wire current, which is the only independent variable controlled here, over nearly two orders of magnitude approximately uniformly on a logarithmic scale to observe the peak fraction of spin flip and its entire range. They started and ended with sufficiently extreme currents that yielded negligible fractions of spin flip. Having reached a nearly zero fraction of spin flip at the highest current might be the reason that they ceased increasing the current further. Further, the calculation of the fraction automatically obviates the requirement for absolute calibration. This data set suggests they designed and executed the experiment with great care.

Frisch and Segrè found that their observation [9] unexpectedly diverges from the Majorana formula (Fig. 1) [10, 11], which was stimulated by the experiment of Frisch and Segrè. The Majorana formula is a variant of the Landau-Zener formula, which is better-known despite the concurrent publications of all four related papers in the same year [12-14]. For a historical comparison of the four papers, please refer to Ref. [15]. Fermi suggested that interaction among atoms could be responsible for the divergence, but atoms were sufficiently sparse to be treated independently [9]. Rabi acknowledged "Professor E. Segrè for discussions on the details of the Frisch and Segrè experiment", recognized the role of the nuclear magnetic moment, and revised the Majorana formula through hyperfine coupling [16]. Rabi's revised formula, however, did not overcome the divergence (Fig. 1).

Multi-stage Stern-Gerlach (Frisch-Segrè) experiments are much more difficult to model than single-stage ones. Multiple stages produce far more nuanced observation because the middle stage can vary the electron spin orientation over a wide range after polarization by the first stage. A correct single-stage theory must pass the more stringent test of the multi-stage experiment. This spin-flip divergence in multi-stage Stern-Gerlach experiments remains unresolved [2]. One may only speculate why the 1933 discrepancy [9] has not been resolved. The seminal paper has not been republished in English, which might have limited its visibility.


Fig. 1. Illustration of the divergence of the Majorana and Rabi formulae from the Frisch-Segrè experimental observation and the convergence of the co-quantum dynamic formula. Details are to be discussed.

Here, a theory, called co-quantum dynamics (CQD) [17], is presented to both provide a collapse mechanism and predict the Frisch-Segrè experimental observation (Fig. 1) [9]. CQD is theoretically verified by reproducing, for electron spin in Stern-Gerlach experiments, the quantum mechanical wave function, density operator, and uncertainty relation as well as, in a recent publication [17], the Schrödinger-Pauli equation. In Methods, CQD is presented in three subsections, including the equations of motion, branching condition, and pre-collapse state function and prediction expression. In Results, Stern-Gerlach experiments in both single and multiple stages are modeled. For flow continuity, lengthy interpretations are postponed to Discussion, and detailed mathematical derivations are presented in Appendices (Supplement Material). Deferred to the last appendices are the CQD derivations of the uncertainty relation, the entangled wave function, and the observation in a two-stage Stern-Gerlach apparatus with a varying angle between the quantization axes.

The following table (Table 1) compares briefly CQD with the representative existing quantum mechanical theories for collapse [18], e.g., the Ghirardi-Rimini-Weber model [19] and
continuous spontaneous localization model [20, 21]. CQD, based on the classical Bloch equation (or its Landau-Lifshitz-Gilbert derivative) and the two postulates, provides a physical instead of phenomenological mechanism for electron spin collapse. In the presence of an external magnetic field, the nuclear magnetic moment is responsible for the collapse of electron spin. The absence of fitting with any adjustable parameters and the high coefficient of determination $R^{2}$ (or high correlation coefficient) led to the small $p$-value ( $p<8 \times 10^{-7}$ ) [22, 23]. In general, fitting with more and more adjustable parameters, one may improve $R^{2}$ towards unity. While $R^{2}$ is not penalized for the number of adjustable parameters used relative to the number of experimental data points available, the $p$-value is. Therefore, one may achieve an arbitrarily high $R^{2}$ at the expense of the $p$-value. The $p$-value is an objective measure of agreement between a theory and the experiment. As a standard definition, the $p$-value quantifies the probability of observing results at least as extreme as the ones observed given that the null hypothesis is true. For stringent discoveries, high-energy physics, for example, requires $p \leqslant 3 \times 10^{-7}$, which corresponds to $5 \sigma$ [24]. The LIGO observation of gravitational waves applied a similar criterion [25]. The agreement of CQD with the experiment is at a similar level as well. While the LIGO observed a chirp signal, which is common in various forms in nature, the Frisch-Segrè experimental data follow an uncommon shape, which is even more unlikely to be matched by random chance. Therefore, the value of $p<8 \times 10^{-7}$ claims a statistical significance that cannot be ignored objectively. The probability that CQD happens to match the experimental observation so well purely by chance is less than one in a million. It is even less likely for an incorrect theory to match an incorrect experiment by chance if one doubts the Frisch-Segrè experimental data. Because the Majorana or Rabi formula, if correct, follows a monotonic trend, it would be difficult to fathom that some experimental imperfections caused the fraction of spin flip to increase at low currents and to decrease at high currents. Matching a theory with the experiment so well without using any adjustable parameters inspires conviction. Further, CQD is corroborated by statistically reproducing exactly the wave function, density operator, and uncertainty relation for electron spin. This corroboration may be considered supporting evidence because an incorrect theory would highly unlikely be able to reproduce so many fundamental aspects of quantum mechanics.

Table 1. Comparison between representative existing quantum mechanical theories and CQD.

|  | Existing theories | Co-quantum dynamics |
| :--- | :--- | :--- |
| Domain | Quantum mechanical | Semiclassical |
| Starting equation | Schrödinger equation | Bloch equation (classical) <br> $[17]$ |
| Cause for collapse | Phenomenological: no <br> physical object identified [18] | Physical: nuclear magnetic <br> moment identified |
| Angular distribution of <br> nuclear magnetic moment | Discrete (quantized); isotropic | Continuous; isotropic or <br> anisotropic |
| Collapse rate | Preset as a constant <br> dimensional rate (1/s) | Scaled dynamically via a <br> dimensionless constant (Eq. <br> $9)$ |
| Measurement uncertainty | Inequality | Equality (Eq. 186), yielding <br> the inequality (Eq. 187) |
| Quantitative prediction of <br> multi-stage Stern-Gerlach <br> (Frisch-Segrè) experiment | Not found yet in the literature <br> except the Majorana or Rabi <br> formulae | Accurately $\left(p<8 \times 10^{-7}\right)$ <br> without scaling or fitting, no <br> parameters are adjusted |

## 2. Methods

## CQD equations of motion

In classical electrodynamics, the motion of a magnetic dipole moment, $\vec{\mu}$, is described by the Bloch equation,

$$
\begin{equation*}
\frac{d \hat{\mu}}{d t}=\gamma \hat{\mu} \times \vec{B}, \tag{1}
\end{equation*}
$$

where caret denotes a unit vector, $t$ time, $\gamma$ the gyromagnetic ratio, and $\vec{B}$ the magnetic flux density. Majorana stated that both the classical and the quantum-mechanical treatments on spin flip require integration of the same differential equations [10, 11]. It is known that the Schrödinger or von Neumann equation for a unitary two-level system can be converted to the Bloch equation or its analog [7, 26, 27].

We now extend the Bloch equation to the Landau-Lifshitz-Gilbert equation [28],

$$
\begin{equation*}
\frac{d \hat{\mu}}{d t}=\gamma \hat{\mu} \times \vec{B}-k_{i} \hat{\mu} \times \frac{d \hat{\mu}}{d t}, \tag{2}
\end{equation*}
$$

where the dimensionless $k_{i}$ is called the induction factor here. Although this equation was originally intended for condensed matter, the underlying physical mechanism for the added term is compatible with CQD (see Paragraph 1 in Discussion). In fact, the author had developed CQD before realizing its connection with the Landau-Lifshitz-Gilbert equation. If $k_{i}=0$, the Bloch equation is recovered.

Henceforth, subscripted $e$ and $n$ denote electron and nucleus, respectively. The default atom, to match the Frisch-Segrè experiment [9], is potassium $\left({ }^{39} \mathrm{~K}\right)$. The scope of the manuscript is limited to potassium in the Stern-Gerlach or Frisch-Segrè experiment.

The torque-averaged magnetic flux densities from $\vec{\mu}_{n}$ and $\vec{\mu}_{e}$ applied on each other are respectively (Appendix 1)

$$
\begin{equation*}
\vec{B}_{n}=\frac{5 \mu_{0}}{16 \pi R^{3}} \vec{\mu}_{n} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{B}_{e}=\frac{5 \mu_{0}}{16 \pi R^{3}} \vec{\mu}_{e}, \tag{4}
\end{equation*}
$$

where $\mu_{0}$ is the vacuum permeability $\left(4 \pi \times 10^{-7} \mathrm{H} / \mathrm{m}\right)$ and $R$ is the van der Waals atomic radius $\left(2.75 \times 10^{-10} \mathrm{~m}\right)$ [29]. Chiefly because the nucleus is more massive, $\mu_{e}\left(9.285 \times 10^{-24} \mathrm{~J} / \mathrm{T}\right) \gg$ $\mu_{n}\left(1.977 \times 10^{-27} \mathrm{~J} / \mathrm{T}\right)$; thus, $B_{e}\left(558.1 \times 10^{-4} \mathrm{~T}\right) \gg B_{n}\left(0.119 \times 10^{-4} \mathrm{~T}\right)$, where $10^{-4} \mathrm{~T}=1$ Gauss.

CQD refers to $\vec{\mu}_{e}$ as the principal quantum and $\vec{\mu}_{n}$ in the same atom as the co-quantum. Postulate 1 states that induction between the electron and the nucleus tends to increase $\left|\theta_{e}-\theta_{n}\right|$, where $\theta$ denotes the polar angle relative to the quantization axis (see Paragraph 1 in Discussion). We (1) apply the Landau-Lifshitz-Gilbert equation to both $\hat{\mu}_{e}$ and $\hat{\mu}_{n}$, (2) express the unit vectors in spherical coordinates, and (3) revise the signs of the induction terms to implement the above postulate, leading to the following CQD equations of motion (Appendix 2):

$$
\begin{equation*}
\dot{\theta}_{e}=-\gamma_{e}\left[B_{y} \cos \phi_{e}+B_{n} \sin \theta_{n} \sin \left(\phi_{n}-\phi_{e}\right)\right]-\operatorname{sgn}\left(\theta_{n}-\theta_{e}\right) k_{i}\left|\dot{\phi}_{e}\right| \sin \theta_{e} \tag{5}
\end{equation*}
$$

$$
\begin{gather*}
\dot{\theta}_{n}=-\gamma_{n}\left[B_{y} \cos \phi_{n}+B_{e} \sin \theta_{e} \sin \left(\phi_{e}-\phi_{n}\right)\right]-\operatorname{sgn}\left(\theta_{e}-\theta_{n}\right) k_{i}\left|\dot{\phi}_{n}\right| \sin \theta_{n},  \tag{6}\\
\dot{\phi}_{e}=-\gamma_{e}\left\{B_{z}+B_{n} \cos \theta_{n}-\cot \theta_{e}\left[B_{y} \sin \phi_{e}+B_{n} \sin \theta_{n} \cos \left(\phi_{n}-\phi_{e}\right)\right]\right\}-\frac{\operatorname{sgn} \dot{\phi}_{e} k_{i}\left|\dot{\theta}_{e}\right|}{\sin \theta_{e}}, \tag{7}
\end{gather*}
$$

and

$$
\begin{equation*}
\dot{\phi}_{n}=-\gamma_{n}\left\{B_{z}+B_{e} \cos \theta_{e}-\cot \theta_{n}\left[B_{y} \sin \phi_{n}+B_{e} \sin \theta_{e} \cos \left(\phi_{e}-\phi_{n}\right)\right]\right\}-\frac{\operatorname{sgn} \dot{\phi}_{n} k_{i}\left|\dot{\theta}_{n}\right|}{\sin \theta_{n}} . \tag{8}
\end{equation*}
$$

Here, $\phi$ denotes the azimuthal angle; $B_{y}$ and $B_{z}$ represent, respectively, the $y$ (axis of the atomic beam) and $z$ components of the external magnetic flux densities; $B_{x}$ is neglected for brevity; sgn denotes the sign function. When $\theta_{e}=0$ or $\pi$, Eq. 7 is replaced with $\dot{\phi}_{e}=0$; when $\theta_{n}=0$ or $\pi$, Eq. 8 is replaced with $\dot{\phi}_{n}=0$. Primarily because the nucleus is more massive again, $\gamma_{e}$ $\left(-1.761 \times 10^{11} \mathrm{rad} \mathrm{Hz} / \mathrm{T}\right)$ in absolute value is four orders of magnitude greater than $\gamma_{n}$ $\left(1.250 \times 10^{7} \mathrm{rad} \mathrm{Hz} / \mathrm{T}\right)$. If $B_{n}=0$ and $k_{i}=0$, Eq. 5 and 7 reduce to the equations shown by Majorana [10, 11].

## CQD branching condition

Postulate 2 states that the polar angle of the co-quantum, $\theta_{n}$, varies negligibly $(\ll \pi)$ during flight in typical Stern-Gerlach experiments, where the duration is too short for the co-quantum to collapse (see Paragraph 2 in Discussion). The external main field, $B_{0}$, along the $z$ axis is usually much stronger than $B_{e}$ and $B_{n}$. While the fast motion of $\hat{\mu}_{e}$ is precession about the main field, the secondary motion is collapse due to the induction term, which yields the following trend from Eq. 5:

$$
\begin{equation*}
\tan \frac{\theta_{e}(t)}{2}=\tan \frac{\theta_{e}(0)}{2} \exp \left[-\operatorname{sgn}\left(\theta_{n}-\theta_{e}\right) k_{i}\left|\Delta \phi_{e}(t)\right|\right] \tag{9}
\end{equation*}
$$

Here, $\Delta \phi_{e}$ denotes the traversed azimuthal angle (i.e., unwrapped phase). If the Larmor frequency of the electron magnetic moment $\omega_{e}$ is constant, we simply have $\Delta \phi_{e}=\omega_{e} t$. As time evolves, $\theta_{e}$ approaches either 0 or $\pi$ according to the following branching condition:

$$
\operatorname{sgn}\left(\theta_{n}-\theta_{e}\right)=\left\{\begin{array}{cc}
1 & \text { if } \theta_{n}>\theta_{e}  \tag{10}\\
0 & \text { if } \theta_{n}=\theta_{e} \\
-1 & \text { else }
\end{array}\right.
$$

Therefore, $\hat{\mu}_{e}$ collapses to either $+z$ or $-z$ while precessing about $B_{0}$, depending on the polar angle of the co-quantum $\theta_{n}$ relative to $\theta_{e}$ (Fig. 2).


Fig. 2. Examples of collapse directions determined by the branching condition in Stern-Gerlach experiments. $B_{0}$ : external main field; $e$ : electron magnetic moment (principal quantum), $\hat{\mu}_{e} ; n$ : nuclear magnetic moment (co-quantum), $\hat{\mu}_{n}$; Short arrows: collapse directions. While $\hat{\mu}_{e}$
precesses about $B_{0}$ right-handedly at its Larmor frequency ( $\omega_{e}=-\gamma_{e} B_{0}$ ), $\hat{\mu}_{n}$ does left-handedly at the Larmor frequency $\left(\omega_{n}=-\gamma_{n} B_{0}\right)$; note that $\left|\omega_{e} / \omega_{n}\right|>10^{4}$. For the same given $\hat{\mu}_{e}$, the collapse direction, down (left panel) or up (right panel), depends on $\hat{\mu}_{n}$ according to the branching condition (Eq. 10). It takes on the order of $N_{c}$ (estimated to be on the order of $\sim 220$ in Results) Larmor cycles to collapse. In typical Stern-Gerlach experiments, it is assumed that $\hat{\mu}_{n}$ does not collapse, i.e., $\theta_{n}$ is approximately constant.

The number of precession cycles required to vary $\tan \left(\theta_{e} / 2\right)$ by a factor of $e$ is given by

$$
\begin{equation*}
N_{c}=\frac{1}{2 \pi k_{i}} \tag{11}
\end{equation*}
$$

regardless of the strength of the external magnetic field. For a constant Larmor frequency, $\omega_{e}$, the collapse time constant is

$$
\begin{equation*}
T_{c}=N_{c} \frac{2 \pi}{\left|\omega_{e}\right|}=\frac{1}{k_{i}\left|\omega_{e}\right|} . \tag{12}
\end{equation*}
$$

## CQD pre-collapse state function and CQD prediction expression

The CQD pre-collapse state function is denoted by $\left|\hat{\mu}_{e} ® \hat{\mu}_{n}\right\rangle$, where the co-quantum, $\hat{\mu}_{n}$, is prefixed with © for clarity. $\left|\hat{\mu}_{e} \Subset \hat{\mu}_{n}\right\rangle$ represents $\hat{\mu}_{e}$ accompanied with $\hat{\mu}_{n}$, both governed by the CQD equations of motion.

The CQD prediction expression for Stern-Gerlach experiments is written as

$$
\begin{equation*}
\left|\hat{\mu}_{e} \odot \hat{\mu}_{n}\right\rangle=C_{+}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right)|+z\rangle+C_{-}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right) \exp \left(i \phi_{e}\right)|-z\rangle \tag{13}
\end{equation*}
$$

The equal sign functions as a right arrow $(\rightarrow)$ because the right side predicts the measurement outcome. A given $\hat{\mu}_{e}$ collapses to either $+\hat{z}$ or $-\hat{z}$ according to the branching condition (Eq. 10). The two real and positive $C$ coefficients take on mutually exclusive binary values while $\exp \left(i \phi_{e}\right)$ captures the phase information. If $\theta_{n}>\theta_{e}$, then $C_{+}=1$ and $C_{-}=0$; if $\theta_{n}<\theta_{e}, C_{+}=0$ and $C_{-}=1$. In either case, $C_{+} \cdot C_{-}=0$ and $C_{+}+C_{-}=1$.

## 3. Results

## Single-stage Stern-Gerlach experiment

To describe the angular distribution of $\hat{\mu}_{e}$ or $\hat{\mu}_{n}$ in an ensemble of atoms, we define the angular probability density function, $p(\theta, \phi)$, as the probability of $\hat{\mu}$ pointing to the vicinity of $(\theta, \phi)$ per unit infinitesimal solid angle, with the following normalization:

$$
\begin{equation*}
\int_{0}^{\pi} \int_{0}^{2 \pi} p(\theta, \phi) \sin \theta d \phi d \theta=1 \tag{14}
\end{equation*}
$$

If the azimuthal distribution is isotropic, the integral reduces to $\int_{0}^{\pi} p(\theta, \phi) 2 \pi \sin \theta d \theta=1$.
The angular distribution of $\hat{\mu}_{n}$ for atoms immediately out of the oven is presumed to be isotropic as given by (Fig. 3, Inset a, dashed circle)

$$
\begin{equation*}
p_{n 0}\left(\theta_{n}, \phi_{n}\right)=\frac{1}{4 \pi} . \tag{15}
\end{equation*}
$$

In a single-stage Stern-Gerlach experiment (Fig. 3, SG1), the probabilities of collapse for a given $\theta_{e}$ are related to the binary coefficients through ensemble averaging of the pre-averaging density operator defined in Appendix 3 (Eq. 70) over $p_{n 0}$. The outcome is summarized as

$$
\begin{equation*}
\left\langle C_{+}\right\rangle_{n}^{2}=\int_{\theta_{e}}^{\pi} p_{n 0} 2 \pi \sin \theta_{n} d \theta_{n}=\cos ^{2} \frac{\theta_{e}}{2} \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle C_{-}\right\rangle_{n}^{2}=\int_{0}^{\theta_{e}} p_{n 0} 2 \pi \sin \theta_{n} d \theta_{n}=\sin ^{2} \frac{\theta_{e}}{2} . \tag{17}
\end{equation*}
$$

The angle brackets, with the subscripts denoting nuclear, represent ensemble averaging with the integration limits determined by the branching condition (Eq. 10). The two probabilities are proportional to the solid angles formed by the down and up sides of the cone shaped by the initial Bloch vector ${ }^{15}$ precessing over one cycle. Each solid angle determines the probability of having the co-quantum on the corresponding side of the cone.


Fig. 3. Multi-stage Stern-Gerlach (SG) experiment conducted by Frisch and Segrè [9]. The atomic beam from the oven is sent through (1) Stage SG1 to collapse $\hat{\mu}_{e}$ (principal quantum), (2) the magnetically shielded inner rotation (IR) chamber to rotate $\hat{\mu}_{e}$, (3) a slit (not shown) to select a branch, and (4) Stage SG2 to measure the fraction of spin flip. The red solid line and filled circle represent the current-carrying wire, and the gray sphere in cutaway view represents magnetic shielding. Inset (a) Angular distributions of $\hat{\mu}_{n}$ (co-quanta) before and after Stage SG1. Inset (b)

Magnetic field lines within the IR chamber; NP: null point, formed by the cancelation of the magnetic field from the wire by the vertical remnant (residual) fringe magnetic field. Here, the vertical distance of the atomic beam from the center of the wire, $z_{a}=1.05 \times 10^{-4} \mathrm{~m}$; the most likely speed of atoms, $v=800 \mathrm{~m} \mathrm{~s}^{-1}$; the uniformly distributed remnant (residual) fringe magnetic flux density, $B_{r}=0.42 \times 10^{-4} \mathrm{~T}$, which is parallel with the $+z$ axis (up in the figure); and the current carried by the wire, $I$, points along the $-x$ axis (into the screen).

From Eq. 16 and 17, the pre-collapse state function (Eq. 13) averages to the following familiar quantum mechanical wave function for a pure state (Appendix 3):

$$
\begin{equation*}
\left|\hat{\mu}_{e}\right\rangle=\cos \frac{\theta_{e}}{2}|+z\rangle+\sin \frac{\theta_{e}}{2} \exp \left(i \phi_{e}\right)|-z\rangle \tag{18}
\end{equation*}
$$

If $\hat{\mu}_{e}$ is also isotropically distributed as

$$
\begin{equation*}
p_{e 0}\left(\theta_{e}, \phi_{e}\right)=\frac{1}{4 \pi}, \tag{19}
\end{equation*}
$$

the probabilities of collapse are predicted by averaging Eq. 16 and 17 over $p_{e 0}$ (Appendix 3):

$$
\begin{equation*}
\left\langle C_{+}\right\rangle_{n, e}^{2}=\int_{0}^{\pi} \cos ^{2} \frac{\theta_{e}}{2} p_{e 0} 2 \pi \sin \theta_{e} d \theta_{e}=\frac{1}{2} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle C_{-}\right\rangle_{n, e}^{2}=\int_{0}^{\pi} \sin ^{2} \frac{\theta_{e}}{2} p_{e 0} 2 \pi \sin \theta_{e} d \theta_{e}=\frac{1}{2} \tag{21}
\end{equation*}
$$

The $e$ subscripts denote electron. The outcomes agree with the familiar quantum mechanical prediction for a maximally mixed state of atoms immediately out of the oven, represented by a density operator (Eq. 87, Appendix 3).

## Multi-stage Stern-Gerlach experiment

In the multi-stage Stern-Gerlach experiment conducted by Frisch and Segrè (Fig. 3) [9], Stage SG1 collapses $\hat{\mu}_{e}$ into two branches. The inner rotation (IR) chamber rotates $\hat{\mu}_{e}$ by an angle of $\alpha_{r}$ using the magnetic field shown in Inset b . A slit (not shown) selects one branch: the $+z$ branch is chosen here. Stage SG2 collapses $\hat{\mu}_{e}$ and measures the fraction of spin flip. Therefore, Stage SG1 serves as a polarizer, the IR chamber a rotator, and Stage SG2 an analyzer.

The probability of spin flip has been predicted [10,11] by quantum mechanics as (see Eq. 17, set $\theta_{e}=\alpha_{r}$ )

$$
\begin{equation*}
W_{\mathrm{qm}}=\left\langle-z \mid \alpha_{r}\right\rangle^{2}=\sin ^{2} \frac{\alpha_{r}}{2}, \tag{22}
\end{equation*}
$$

which leads to the following Majorana formula (Appendix 4, Eq. 117) [10, 11]:

$$
\begin{equation*}
W_{m}=\exp \left(-\frac{\pi z_{a}}{2 v}\left|\gamma_{e}\right| B_{y}\right) \tag{23}
\end{equation*}
$$

Here, $z_{a}$ is the vertical distance of the atomic beam from the center of the wire, and $v$ is the most likely speed of the atoms. The spin flip is because $B_{z}$ vanishes and reverses its sign near the null point (Fig. 3, Inset b). Because $B_{y}$ is inversely proportional to the current carried by the wire, $I$ (Eq. 92 and 94 in Appendix 4), the Majorana formula predicts a probability of spin flip approaching $100 \%$ with increasing currents (Fig. 4, Curve m), i.e., as $B_{y} \rightarrow 0, W_{m} \rightarrow 1$; yet, the experimental
outcome decreases to nearly zero after peaking at $31 \%$ (Fig. 4, circles) [9]. Consequently, $W_{m}$ yields a negative coefficient of determination $\left(R^{2}\right)$. Using the dimensionless adiabaticity parameter $k_{m}$ (Eq. 103 in Appendix 4), one can express the above equation concisely as $W_{m}=$ $\exp \left(-\pi k_{m} / 2\right)$ (Eq. 116). Rabi revised the Majorana formula to $W_{m}^{1 / 4} / 4$ [16], which, however, overestimates the starting points, underestimates the peak, and continues to diverge thereafter; as a result, the $R^{2}$ remains negative (Fig. 1).


Fig. 4. Fraction of spin flip versus wire current. The down arrow points to the current where $B_{y}^{\prime}=$ $B_{n} \sin \left\langle\theta_{n}\right\rangle$ or $k_{0}=k_{1}$ to separate the low- and high-current regions. Curves m and $1-4$ represent $W_{m}$ and $W_{1}-W_{4}$, respectively. While $W_{m}$ diverges from the experiment with a negative $R^{2}, W_{3}$ matches the low-current experimental observation in absolute units without fitting with $R^{2}=$ 0.9495 ; further, $W_{4}$ matches the entire observation with improved $R^{2}=0.9787$ and $p<$ $8 \times 10^{-7}$. No adjustable or free parameters are used.

In CQD, Stage SG1 varies $\theta_{n}$ negligibly according to Postulate 2. However, polarization selection by the slit reshapes the co-quantum angular distribution from the original isotropic $p_{n 0}$ (Eq. 15) to

$$
\begin{equation*}
p_{n 1}\left(\theta_{n}, \phi_{n}\right)=p_{n 0}\left(\theta_{n}, \phi_{n}\right) \cdot 2 \int_{0}^{\theta_{n}} p_{e 0} 2 \pi \sin \theta_{e} d \theta_{e}=\frac{1-\cos \left(\theta_{n}\right)}{4 \pi} . \tag{24}
\end{equation*}
$$

Here, the pre-factor 2 compensates for the overall slit rejection of the opposite polarization (Eq. 20), $p_{e 0}$ is given by Eq. 19, and the integration limits are based on the branching condition (Eq. 10). Because atoms with smaller $\theta_{n}$ are deflected to the blocked $-z$ branch with greater probabilities, $p_{n 1}$ forms a heart shape (Fig. 3, Inset a, solid line; Paragraph 3 in Discussion).

The heart shape is assumed to be approximately maintained throughout the inner rotation chamber owing to the extension of Postulate 2 (see Paragraph 2 in Discussion). The co-quanta engender the following four effects on the principal quanta.

First, the probability of spin flip is derived by ensemble averaging over $p_{n 1}$ instead of $p_{n 0}$ (Eq. 89 with $\theta_{e}=\alpha_{r}$ in Appendix 3):

$$
\begin{equation*}
W_{\mathrm{cqd}}=\left\langle-z \mid \alpha_{r}\right\rangle^{2}=\int_{0}^{\alpha_{r}} p_{n 1} 2 \pi \sin \theta_{n} d \theta_{n}=\sin ^{4}\left(\frac{\alpha_{r}}{2}\right), \tag{25}
\end{equation*}
$$

which equals $W_{\mathrm{qm}}^{2}$ (Eq. 22). As shown by Curve 1 in Fig. 4, simply squaring $W_{m}$ (Eq. 23) already brings the solution much closer to the observation at low currents, but with an overcorrection near $I=0.03$ A. This squaring effect evolves the probability of spin flip from $W_{m}$ to

$$
\begin{equation*}
W_{1}=W_{m}^{2}=\exp \left(-\frac{\pi z_{a}}{v}\left|\gamma_{e}\right| B_{y}\right) \tag{26}
\end{equation*}
$$

where $B_{y}$ is computed from the remnant (residual) fringe magnetic flux density, $B_{r}$, using Eq. 92 and 94 in Appendix 4. Using the dimensionless adiabaticity parameter $k_{m}$ (Eq. 103), one can express the above equation concisely as $W_{1}=\exp \left(-\pi k_{m}\right)$ (see Eq. 154 in Appendix 5).

Second, the $z$ component of $\vec{B}_{n}$ (Eq. 3), represented by $B_{n} \cos \left\langle\theta_{n}\right\rangle$, offsets the upward $B_{r}$. We substitute $B_{r}+B_{n} \cos \left\langle\theta_{n}\right\rangle$ (Eq. 119 in Appendix 5) for $B_{r}$ to update $B_{y}$ to $B_{y}^{\prime}$ (Eq. 122). The heart shape (Eq. 24) yields $\left\langle\theta_{n}\right\rangle=5 \pi / 8$ (Eq. 118). The magnitude of $B_{n} \cos \left\langle\theta_{n}\right\rangle=-0.045 \times$ $10^{-4} \mathrm{~T}$ exceeds $10 \%$ of $B_{r}\left(0.42 \times 10^{-4} \mathrm{~T}\right)$, producing an appreciable remnant-alteration effect. As shown by Curve 2 in Fig. 4, the corrected curve passes through the first two data circles and grazes the third one. If the co-quantum distribution were isotropic, $\left\langle\theta_{n}\right\rangle$ would be $\pi / 2 ; B_{n} \cos \left\langle\theta_{n}\right\rangle$ would vanish, so would the remnant-alteration effect. Effect 2 evolves $W_{1}$ to (see Eq. 136)

$$
\begin{equation*}
W_{2}=\exp \left(-\frac{\pi z_{a}}{v}\left|\gamma_{e}\right| B_{y}^{\prime}\right), \tag{27}
\end{equation*}
$$

where $B_{y}^{\prime}$, however, is computed using $B_{r}+B_{n} \cos \left\langle\theta_{n}\right\rangle$ instead of $B_{r}$. Using the dimensionless adiabaticity parameters $k_{0}$ (Eq. 125), one can express the above equation concisely as $W_{2}=$ $\exp \left(-\pi k_{0}\right)$ (see Eq. 153).

Third, the co-quanta saturate the rotation. As shown by Eq. 27, $W_{2}$ increases with decreasing $B_{y}^{\prime}$. However, the weakness of $B_{y}^{\prime}$ is spoiled by the transverse $(x y)$ component of $\vec{B}_{n}$, denoted by $B_{n} \sin \left\langle\theta_{n}\right\rangle$. Substitution of $\sqrt{B_{y}^{\prime 2}+\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)^{2}}$ for $B_{y}^{\prime}$ (see Eq. 141) evolves $W_{2}$ to

$$
\begin{equation*}
W_{3}=\exp \left(-\frac{\pi z_{a}}{v}\left|\gamma_{e}\right| \sqrt{B_{y}^{\prime 2}+\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)^{2}}\right) . \tag{28}
\end{equation*}
$$

Using the dimensionless adiabaticity parameters $k_{0}$ (Eq. 125) and $k_{1}$ (Eq. 126), one can express the above equation concisely as $W_{3}=\exp \left(-\pi \sqrt{k_{0}^{2}+k_{0} k_{1}}\right)$ (see Eq. 152).

As shown by Curve 3 in Fig. 4, this rotation-saturation effect clamps the overshoot in Curve 2. The clamped curve passes through the first four data circles. The current is divided into two regions at 0.067 A , where $B_{y}^{\prime}=B_{n} \sin \left\langle\theta_{n}\right\rangle=0.11 \times 10^{-4} \mathrm{~T}$. At low currents before the fourth data point $\left(I=0.05 \mathrm{~A}\right.$ and $\left.B_{y}^{\prime}=0.15 \times 10^{-4} \mathrm{~T}\right), B_{y}^{\prime}$ is greater than $B_{n} \sin \left\langle\theta_{n}\right\rangle$; at high currents, conversely, $B_{n} \sin \left\langle\theta_{n}\right\rangle$ becomes dominant and saturates the curve. If the co-quantum distribution
were isotropic, then $\left\langle\theta_{n}\right\rangle=\pi / 2$; consequently, both the squaring and remnant-alteration effects (effects 1 and 2 ) would vanish. In this case, the rotation-saturation effect (effect 3 ) alone could not bring the Majorana solution down sufficiently in the low-current region because as the current decreases $B_{y}^{\prime}$ increasingly overpowers $B_{n}$; thus, the effect of the co-quanta would become negligible.

Combining the three effects, CQD accurately predicts the low-current observation in absolute units without fitting (i.e., no parameters are adjusted). The coefficient of determination $R^{2}$ for the low-current regime reaches 0.9495 as computed using the natural logarithm of the fractions of flip to suppress the exponential dependence (Eq. 28). Therefore, effecting the three modifications to the Majorana formula has already shown evidence for the existence of both the co-quantum and the derived heart-shaped distribution.

Fourth, in the high-current regime, the precession of $\vec{B}_{n}$ generates substantial nuclearresonant rotation, due to precession resonance between $\vec{\mu}_{e}$ and $\vec{\mu}_{n}$ when their Larmor frequencies are matched (see Appendix 5 for details). This effect evolves $W_{3}$ to (Eq. 167 in Appendix 5)

$$
\begin{equation*}
W_{4}=W_{3} \exp \left(-c_{r 1} I^{3}\right) \tag{29}
\end{equation*}
$$

where the resonant-rotation coefficient, $c_{r 1}$, is given by Eq. 163. The fraction of spin flip peaks near $I=0.1 \mathrm{~A}$, where $B_{y}^{\prime}=0.074 \times 10^{-4} \mathrm{~T}$, comparable to but less than $B_{n} \sin \left\langle\theta_{n}\right\rangle=0.11 \times$ $10^{-4} \mathrm{~T}$. As shown by Curve 4 in Fig. 4, this effect increases with the current and bends down Curve 3. At the maximum current $(I=0.5 \mathrm{~A}), B_{y}^{\prime}=0.015 \times 10^{-4} \mathrm{~T}$, far less than $B_{n} \sin \left\langle\theta_{n}\right\rangle$; the fraction of spin flip decreases to nearly zero. Expression of the above equation based on dimensionless parameters is discussed below Eq. 150. Using the dimensionless adiabaticity parameters $k_{0}$ (Eq. 125) and $k_{1}$ (Eq. 126) as well as $f_{r 1}$ (Eq. 146), one can express the above equation concisely as $W_{4}=\exp \left(-\pi \sqrt{k_{0}^{2}+k_{0} k_{1}}-\frac{1}{2}\left[\pi k_{1}\right]^{2} f_{r 1}\right)$ (see Eq. 151), where $f_{r 1}$ denotes the fraction of the Larmor period of the nuclear magnetic moment precessed during the effective flight path-length for nuclear-resonant rotation.

Combining all four effects, CQD accurately predicts the experimental observation in absolute units without fitting (Fig. 4, Curve 4) over the entire domain; $R^{2}$ is computed to be 0.9787 using the natural logarithm of the fractions of flip to suppress the exponential dependence (Eq. 29). Under the null hypothesis that the theoretical prediction is uncorrelated with the observation, we estimate the $p$-value to be $<8 \times 10^{-7}$ (Function regress or corr, MATLAB, MathWorks) [22, $23]$. Such a small $p$-value further objectively confirms the existence of both the co-quantum and the derived heart-shaped distribution. In comparison, without taking the logarithm of the fractions of flip, $R^{2}$ is computed to be 0.9621 .

Thus far, we have set the induction factor $k_{i}=0$ for the flight in the inner rotation chamber, owing to the low field (Appendix 5). Including $k_{i}$ yields the following combined probability of spin flip (Appendix 5, Eq. 160):

$$
\begin{equation*}
W_{\mathrm{cqd}}=\exp \left[-\sqrt{\left(c_{r 0} / I\right)^{2}+c_{r s}^{2}}-c_{r 1} I^{3}-c_{r i} I\right] \tag{30}
\end{equation*}
$$

where $c_{r 0}, c_{r s}, c_{r 1}$, and $c_{r i}$ represent null-point rotation, rotation saturation, nuclear-resonant rotation, and induction rotation, respectively. The current, $I$, controls the external magnetic field in the inner rotation chamber. Taken from Frisch and Segrè [9], the only device-specific parameters
for the predictions include $B_{r}\left(0.42 \times 10^{-4} \mathrm{~T}\right), z_{a}\left(1.05 \times 10^{-4} \mathrm{~m}\right)$, and $v\left(800 \mathrm{~m} \mathrm{~s}^{-1}\right)$. The theoretical predictions from Eq. 161-163 without adjusting any parameters are $c_{r 0}=0.054 \mathrm{~A}, c_{r s}$ $=0.80$, and $c_{r 1}=48 \mathrm{~A}^{-3}$. Substitution of these coefficients into Eq. 30 with $c_{r i}=0$ produces Curve 4 in Fig. 4, where no free parameters are used.

Despite its small contribution in the inner rotation chamber, the induction factor is estimated for its order of magnitude. While holding all three other parameters constant at the predicted values, fitting $W_{\text {cqd }}$ (Eq. 30) for the experimental data in Fig. 4 yields $c_{r i} \sim 0.57 \mathrm{~A}$. Substitution into Eq. 164 produces $k_{i} \sim 7.4 \times 10^{-4}$. Further substitution into Eq. 11 concludes that electron-spin collapse takes on the order of $N_{c} \sim 220$ precession cycles.

## 4. Discussion

CQD postulates that the electron and nuclear magnetic moments in an external field $B_{0}$ along $z$ repel in the polar direction, which results in a revision to the sign of the induction term in the Landau-Lifshitz-Gilbert equation. Whereas precession is governed by the terms from the Bloch equation, collapse is modeled by the revised induction term. If $k_{i}=0$, the equation of motion reduces to the Bloch or equivalent Schrödinger equation [7,10,11, 26, 27], which does not model collapse [30]. While precession is the dominant motion, collapse is secondary but concurrent. Although the exact mechanism for the repulsion is to be investigated, a conjecture is diamagnetism extended from orbital to spin motions. Diamagnetic magnetization, a weak but universal induction effect on all atoms, causes repulsion [31, 32]. The relativistic momentum density in the Dirac wave field shows that the magnetic moment of an electron can be attributed to a circulating flow of electric charge (Eq. 34 in Appendix 1), similar to that in orbital motions [33]. Therefore, it is conceivable that diamagnetism applies to spin as well. In the laboratory reference frame, as $\vec{\mu}_{e}$ and $\vec{\mu}_{n}$ precess in opposite directions, each azimuthal encounter may be viewed as a "collision", causing repulsion. Because induction is related to relative motion, the induced field on the electron may be written as $\vec{B}_{i} \propto d\left(\hat{\mu}_{e}-\hat{\mu}_{n}\right) / d t$, and the corresponding induced torque is $\vec{\tau}_{i} \propto \hat{\mu}_{e} \times \vec{B}_{i}$. If $\hat{\mu}_{e} \times d \hat{\mu}_{n} / d t$ averages out, the average induced torque becomes $\hat{\mu}_{e} \times d \hat{\mu}_{e} / d t$, matching the induction term in the Landau-Lifshitz-Gilbert equation. As $\hat{\mu}_{e}$ nears either up or down, the average induced torque approaches zero, providing stability. In the rotating reference frame that rotates at $\omega_{e}$, the external $B_{0}$ vanishes, $\vec{\mu}_{e}$ becomes azimuthally stationary [34]; the rapidly precessing $\vec{\mu}_{n}$ forms in the time-average sense a cone-shaped magnet, which repels $\vec{\mu}_{e}$ towards $\pm z$. The sign function in the induction terms in the equations of motion is the key difference from the standard Landau-Lifshitz-Gilbert equation and is central to CQD. While standard damping always leads to a lower-energy state, collapse due to the co-quantum can reach a state of either higher or lower energy in the presence of an external magnetic field, according to the branching condition, which agrees with the Stern-Gerlach experimental observation. Numerical solutions to the CQD equations of motion, to be reported separately, have illustrated collapse with the induction term and none without. This postulate is consistent with the Pauli exclusion principle for two identical fermions, where the two magnetic moments repel towards anti-alignment. Therefore, one may regard this postulate as an extension to the Pauli exclusion principle. Note that while diamagnetism explains the collapse term, paramagnetism is expected to perturb the precession term slightly, which is neglected here.

CQD also postulates that the polar angle of $\vec{\mu}_{n}$ in flight varies negligibly. Because the nuclear Larmor frequency is four orders of magnitude smaller (i.e., $\left|\omega_{n}\right| \ll\left|\omega_{e}\right|$ ), nuclear spin
collapses much more slowly than electron spin. Because no data on the collapse rates have been found in the literature, we reference the $T_{1}$ relaxation times. Typical $T_{1}$ relaxation times in electron paramagnetic resonance are on the $\mu \mathrm{s}$ scale [35], consistent with the previous estimation of the collapse time scale of $\vec{\mu}_{e}$ [5]. In contrast, typical $T_{1}$ relaxation times in gas-phase nuclear magnetic resonance are on the ms scale [36], indicating the order-of-magnitude collapse time of $\vec{\mu}_{n}$. In a typical Stern-Gerlach experiment [5,37], the main external field $B_{0}$ along $z$ is at least 0.3 T ( $B_{0}>$ $B_{e} \gg B_{n}$, the Paschen-Back regime [16]), the length of the main field is $\sim 35 \mathrm{~mm}$, and the most likely atomic speed $v$ is $\sim 800 \mathrm{~m} \mathrm{~s}^{-1}$. Consequently, the flight time through the main field is only $\sim 44 \mu \mathrm{~s}$, which is long enough for $\vec{\mu}_{e}$ to collapse but too short for $\vec{\mu}_{n}$ to collapse. In fact, the fringe field on the source side of the main field collapses $\vec{\mu}_{e}$ [2]. Besides the two distinct collapse branches due to the quantization of $\vec{\mu}_{e}$, no additional branches due to the quantization of $\vec{\mu}_{n}$ have been observed by Frisch and Segrè [9] despite the prediction of up to eight branches total [38]. For $N_{c} \sim 220$ (Eq. 11) estimated from the Frisch-Segrè experimental data shown in Fig. 4, the collapse time constants ( $T_{c}$, Eq. 12 and its nuclear counterpart) at the main field strength are computed to be $\sim 3 \times 10^{-8}$ and $\sim 4 \times 10^{-4}$ s for $\vec{\mu}_{e}$ and $\vec{\mu}_{n}$, respectively, which are consistent with the abovementioned corresponding $T_{1}$ relaxation times in orders of magnitude [35] [36]. This postulate, extended to the weaker-field inner rotation chamber, is consistent with the selection rule for observing an electron-spin-resonance transition, stating that the magnetic quantum number of the nuclear spin remains constant (i.e., $\Delta m_{I}=0$ ) [39]. The selection rule was also a major basis for Rabi's revision to the Majorana formula [16].

The heart-shaped $p_{n 1}$ in Eq. 24 (Fig. 3, Inset a) can be understood in two ways. First, the integral can be perceived as the expected transmittance through Stage SG1 for a given $\theta_{n}$. All principal quanta with $\theta_{e}<\theta_{n}$ collapse to $+z$, and the atoms propagate through the slit further; otherwise, the atoms are blocked by the slit. The greater the $\theta_{n}$ is, the greater the transmittance, proportional to the solid angle formed by the cone having a half angle of $\theta_{n}$ (Fig. 5). Second, one may examine how much principal quanta at the source around each $\theta_{e}$ within $d \theta_{e}$ contribute to $p_{n 1}$. For $\theta_{e}=0$, the contribution forms a perfect spherical distribution of co-quanta because coquanta in any direction can reach the second stage. For $0<\theta_{e}<\pi$, the contribution forms a truncated sphere with the cone of $\theta_{n}<\theta_{e}$ removed because co-quanta in this range have collapsed the principal quanta to the blocked branch. For $\theta_{e}=\pi$, the contribution vanishes because the principal quanta are always in the blocked branch. Integrating these (truncated) spheres form the final heart shape. Conversely, the co-quantum angular distribution for the opposite branch is an inverted heart shape. Average the two complementary shapes recovers the original isotropic $p_{n 0}$.


Fig. 5. Illustration of the cone of $\hat{\mu}_{n}$ formed by precession around the external main field, $B_{0} . n$ : nuclear magnetic moment (co-quantum), $\hat{\mu}_{n}$. Any electron magnetic moment (principal quantum), $\hat{\mu}_{e}$, precessing around $B_{0}$ within the cone collapses up, whereas $\hat{\mu}_{e}$ precessing outside the cone
collapses down. For a given $\theta_{n}$, the probability for the atom from the oven to reach the up branch in the single-stage Stern-Gerlach experiment is proportional to the solid angle of the cone.

A key reason for the agreement between CQD and the Frisch-Segrè experimental observation is that the angular distribution of the co-quantum (i.e., the nuclear magnetic moment) is changed from an isotropic shape (Eq. 15) to a continuous heart shape (Eq. 24) due to the polarization. The subsequent effects are illustrated using the evolution of the curves in Fig. 4. As more effects are included, the model becomes more and more accurate while all parameters were given (i.e., no parameters were tuned to fit the experimental data). If the heart shape were incorrect, the agreement would be completely off. In comparison, the Majorana or Landau-Zener formula neglected the nuclear magnetic moment altogether, and Rabi used an isotropic angular distribution instead of the heart shape [16]. Note that as the wire current approaches infinity, Rabi's formula predicts a maximum of $\frac{1}{2 I+1}=\frac{1}{4}$, which is well below the experimental peak of $31 \%$ (Fig. 1); here, $I=\frac{3}{2}$ denotes the nuclear spin number for potassium-39. Further, Rabi's standard hyperfine coupling does not contain the induction terms in CQD and hence does not model collapse. Also, the torque-averaged fields provide greater agreement than the self-averaged fields (see Appendix 1).

Quantum mechanics, celebrated for its countless triumphs, still pose mysteries as discussed insightfully in recent literature [30, 40-43]. The Copenhagen interpretation construes that an electron spin is simultaneously in both eigenstates and collapses statistically upon measurement to either [7]. The collapse is not modeled by the original Schrödinger equation but stated separately as a measurement postulate [30]. Debatable inconsistency has been found in thought experiments, such as "Schrödinger's cat" [44-46].

CQD potentially offers new insight. If co-quanta are isotropically distributed, CQD has been verified with quantum mechanics by exactly reproducing the wave function and the density operator (Appendix 3) as well as the uncertainty relation (Appendix 6) and entangled wave function (Appendix 7). The probabilities of reaching the two eigenstates split according to $\cos ^{2} \frac{\theta_{e}}{2}: \sin ^{2} \frac{\theta_{e}}{2}$; the wave function is reproduced in Eq. 18. However, if the co-quanta have, for example, a heart-shaped distribution (Eq. 24), the split becomes $1-\sin ^{4} \frac{\theta_{e}}{2}: \sin ^{4} \frac{\theta_{e}}{2}$ (Eq. 88 and 89 in Appendix 3); the wave function is revised accordingly (Eq. 90). The density operator is found to originate from a pre-averaging counterpart with independent realizations (Appendix 3). The measurement uncertainty product, explained by co-quanta, depends on the initial phase of the principal quanta and the measurement sequence, as shown by the uncertainty equality (Eq. 186), which leads to the familiar quantum mechanical inequality (Eq. 187). CQD has also enabled the derivation from the classical Bloch equation to the quantum Schrödinger-Pauli equation [17], while the latter has thus far been treated as a postulate.

CQD can be further tested with atoms having nuclear spins of $0\left(\mu_{n}=0\right)$, which may collapse differently in Stern-Gerlach experiments. A natural question is whether higher-order nuclear multipoles could serve as co-quanta. Examples include ${ }^{38 \mathrm{ml}} \mathrm{K},{ }^{50} \mathrm{~K},{ }^{94} \mathrm{Ag}$, and ${ }^{130} \mathrm{Ag}$, which are isotopes of the stable ${ }^{39} \mathrm{~K},{ }^{107} \mathrm{Ag}$, and ${ }^{109} \mathrm{Ag}$. Unfortunately, these isotopes have short lifetimes ranging from 100s to 10 s of ms . Note that free electrons have not been used owing to the Lorentz force and orbital magnetic moment.

Since the submission of this manuscript, our team has produced several new manuscripts to support this work. Titimbo et al. numerically modeled the Frisch-Segrè experiment using CQD via the Bloch equation [47], whereas He et al. numerically modeled the experiment using CQD via the Schrödinger equation [48]. Both works have numerically confirmed the analytical solution presented here and the equivalence between the the Bloch equation and the Schrödinger equation stated by Majorana. Interestingly, Majorana wrote the "Bloch" equation [10, 11] fourteen years before Bloch published his eponymous equation [49]. The author recently derived from the Bloch equation, which Bloch intended for macroscopic magnetization instead of individual nuclear magnetic moments [49], to the Schrödinger or Schrödinger-Pauli equation [17]. Kahraman et al. demonstrated that the standard existing treatment of hyperfine interaction, consistent with the Breit-Rabi formula [38], cannot model the Frisch-Segrè experiment accurately but can be improved by incorporating CQD features [50]. The treatment also does not agree with the Rabi formula [16].

While no alternative theory, to the best of our knowledge, matches the Frisch-Segrè experiment, a recent multi-stage Stern-Gerlach experiment on superatomic icosahedral cageclusters $\mathrm{Mn} @ \mathrm{Sn}_{12}$ also reveals discrepancy of the Landau-Zener formula from experimental observation [51].

## 5. Conclusions

CQD, based on the sign-modified Landau-Lifshitz-Gilbert equation, provides a plausible collapse mechanism for electron spin in Stern-Gerlach experiments. CQD models both spin evolution and collapse by the same equations of motion. With an anisotropic angular distribution of co-quanta, CQD revises the wave function and accurately predicts the Frisch-Segrè experimental observation in absolute units without fitting with adjustable parameter, achieving $p<8 \times 10^{-7}$-an objective statistical indication that reflects both correlation and degrees of freedom. Therefore, it is extremely unlikely that CQD happens to match the experimental observation so well by sheer chance. Further, with an isotropic angular distribution of co-quanta, CQD is theoretically corroborated by quantum mechanics. Both the strong experimental evidence and the exact quantitative agreement with quantum mechanics in diverse forms collectively support CQD. Like statistical mechanics [52], which uses molecular properties to predict macroscopic properties by ensemble averaging, CQD reproduces quantum mechanical properties by ensemble averaging over co-quanta (Appendix 3). If orthodox quantum mechanics is incomplete [44], CQD may stimulate development for a complete theory.

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Data availability statement
The data that support the findings of this study are available upon request from the authors.

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## Supplementary Material

## Multi-stage Stern-Gerlach experiment modeled

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## Appendix 1. Derivation of torque-averaged fields

Given the focus of the Landau-Lifshitz-Gilbert equation on torque, we derive the torque-averaged magnetic flux densities applied on the electron and the nucleus by each other.

In relativistic quantum mechanics, the momentum density in the Dirac wave field is given by [33]

$$
\begin{equation*}
\vec{G}=\frac{\hbar}{2 i}\left[\psi^{\dagger} \nabla \psi-\left(\nabla \psi^{\dagger}\right) \psi\right]+\frac{\hbar}{4} \nabla \times\left(\psi^{\dagger} \vec{\sigma}_{4 \times 4} \psi\right) \tag{31}
\end{equation*}
$$

Here, $\psi$ denotes the spatial wave function, $\psi_{s}$, multiplied by the spinor $w^{1}(0)=(1,0,0,0)^{\dagger}$; $\vec{\sigma}_{4 \times 4}=\sigma_{1} \hat{x}+\sigma_{2} \hat{y}+\sigma_{3} \hat{z}$, where $\sigma_{1}=-i \alpha_{2} \alpha_{3}, \sigma_{2}=-i \alpha_{3} \alpha_{1}, \sigma_{3}=-i \alpha_{1} \alpha_{2}$; matrices $\alpha_{1}, \alpha_{2}$, and $\alpha_{3}$ are from the Dirac equation [53]:

$$
\alpha_{1}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1  \tag{32}\\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right), \alpha_{2}=\left(\begin{array}{cccc}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
i & 0 & 0 & 0
\end{array}\right) \text {, and } \alpha_{3}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array}\right) .
$$

While the first term on the right side of Eq. 31 is attributed to the translational motion of the electron, the second term is associated with circulating flow of energy [33].

Following Ohanian [33], an $s$ orbital wave function is considered; $\psi_{s}^{\dagger} \psi_{s}$ is set to the Gaussian distribution,

$$
\begin{equation*}
\rho(r)=\left(\frac{1}{\pi a_{0}^{2}}\right)^{\frac{3}{2}} \exp \left(-\frac{r^{2}}{a_{0}^{2}}\right), \tag{33}
\end{equation*}
$$

where $r$ denotes the radial coordinate. The average radius, $\frac{2 a_{0}}{\sqrt{\pi}}$, is set to the van der Waals atomic radius, $R$.

While the first term in Eq. 31 vanishes, the second term becomes

$$
\begin{equation*}
\vec{G}=\frac{\hbar}{2 a_{0}^{2}} \rho \hat{z} \times \vec{r} . \tag{34}
\end{equation*}
$$

The differential element of the magnetic dipole moment is [33]

$$
\begin{equation*}
d \vec{m}_{e}(\vec{r})=\gamma_{e} \vec{r} \times\left(\frac{\hbar}{4} \nabla \times\left(\psi^{\dagger} \gamma^{0} \vec{\sigma}_{4 \times 4} \psi\right)\right) d^{3} \vec{r} \tag{35}
\end{equation*}
$$

From

$$
\gamma^{0}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{36}\\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right),
$$

we reach

$$
\begin{equation*}
d \vec{m}_{e}(\vec{r})=\gamma_{e} \vec{r} \times \vec{G} d^{3} \vec{r} \tag{37}
\end{equation*}
$$

The field at location $\vec{r}$ from $\vec{\mu}_{n}$ is given by $[31,32]$

$$
\begin{equation*}
\vec{B}(\vec{r})=\frac{\mu_{0}}{4 \pi r^{3}}\left[3\left(\vec{\mu}_{n} \cdot \hat{r}\right) \hat{r}-\vec{\mu}_{n}\right]+\frac{2 \mu_{0}}{3} \vec{\mu}_{n} \delta(\vec{r}) . \tag{38}
\end{equation*}
$$

The differential element of the torque from $\vec{\mu}_{n}$ is

$$
\begin{equation*}
d \vec{\tau}_{n}(\vec{r})=d \vec{m}_{e}(\vec{r}) \times \vec{B}(\vec{r}) . \tag{39}
\end{equation*}
$$

Volumetric integration yields

$$
\begin{equation*}
\vec{m}_{e}(\vec{r})=\int d \vec{m}_{e}=\gamma_{e} \frac{\hbar}{2} \hat{z}, \tag{40}
\end{equation*}
$$

which equals $\vec{\mu}_{e}$, and

$$
\begin{equation*}
\vec{\tau}_{n}=\int d \vec{\tau}_{n}=\vec{\mu}_{e} \times\left(\frac{4 \mu_{0}}{3 \pi^{3} R^{3}} \vec{\mu}_{n}\right) . \tag{41}
\end{equation*}
$$

Therefore, the torque-averaged $B$ field from $\vec{\mu}_{n}$ applied on the electron is given by

$$
\begin{equation*}
\vec{B}_{n}=\frac{4 \mu_{0}}{3 \pi^{3} R^{3}} \vec{\mu}_{n} . \tag{42}
\end{equation*}
$$

Now, we switch to a classical approach. On the time scale pertinent to the precession cycle, we model the much faster motion of the $s$ valence electron with the probability density, $\rho$.

The current density at position $\vec{r}$ is given by

$$
\begin{equation*}
\vec{\jmath}=-e \rho \vec{\omega} \times \vec{r}, \tag{43}
\end{equation*}
$$

where $-e$ denotes the electron charge and $\vec{\omega}$ denotes the angular velocity.
The differential element of the magnetic dipole moment is

$$
\begin{equation*}
d \vec{m}_{e}(\vec{r})=\frac{1}{2} \vec{r} \times \vec{\jmath} d^{3} \vec{r} \tag{44}
\end{equation*}
$$

The differential element of the torque is (Eq. 39)

$$
\begin{equation*}
d \vec{\tau}_{n}(\vec{r})=d \vec{m}_{e}(\vec{r}) \times \vec{B}(\vec{r}) . \tag{45}
\end{equation*}
$$

Three distributions of $\rho$ are considered.
First, $\rho$ is set to the Gaussian distribution given by Eq. 33. Volumetric integration produces

$$
\begin{equation*}
\vec{m}_{e}=\int d \vec{m}_{e}=-\frac{1}{2} e a_{0}^{2} \vec{\omega}=-\frac{\pi}{8} e R^{2} \vec{\omega} \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{\tau}_{n}=\int d \vec{\tau}_{n}=\vec{m}_{e} \times\left(\frac{4 \mu_{0}}{3 \pi^{3} R^{3}} \vec{\mu}_{n}\right) \tag{47}
\end{equation*}
$$

Averaging yields

$$
\begin{equation*}
\left\langle\vec{\tau}_{n}\right\rangle=\left\langle\vec{m}_{e}\right\rangle \times\left(\frac{4 \mu_{0}}{3 \pi^{3} R^{3}} \vec{\mu}_{n}\right) . \tag{48}
\end{equation*}
$$

For an $s$ valence electron, both the orbital angular momentum and the orbital magnetic moment vanish; accordingly, we have $\left\langle\vec{m}_{e}\right\rangle=\vec{\mu}_{e}$, which is due to spin only. Consequently,

$$
\begin{equation*}
\left\langle\vec{\tau}_{n}\right\rangle=\vec{\mu}_{e} \times\left(\frac{4 \mu_{0}}{3 \pi^{3} R^{3}} \vec{\mu}_{n}\right) \tag{49}
\end{equation*}
$$

Therefore, we reach

$$
\begin{equation*}
\vec{B}_{n}=\frac{4 \mu_{0}}{3 \pi^{3} R^{3}} \vec{\mu}_{n}, \tag{50}
\end{equation*}
$$

which agrees with the relativistic quantum mechanical solution (Eq. 42).
Second, using the tabulated approximate $4 s$ wave function, $\psi_{s}$, for potassium (Table S1) [54] based on Hartree's self-consistent field [55], we numerically reached

$$
\begin{equation*}
\vec{B}_{n}=\frac{0.138 \mu_{0}}{\pi R^{3}} \vec{\mu}_{n}, \tag{51}
\end{equation*}
$$

where the average radius is set to $R$. Coincidentally, this solution differs from the Gaussian solution (Eq. 50 ) by only $2 \%$.

Table S1. Normalized $P(4 s)$ for potassium. $\psi_{s}(0)=\sqrt{9.76 /(4 \pi)}$ and $\psi_{s}(r)=P /\left(\sqrt{4 \pi a_{0}} r\right)$ for $r>0$. [54]

| $\boldsymbol{r} / \boldsymbol{a}_{\mathbf{0}}$ | $\boldsymbol{P}$ | $\boldsymbol{r} / \boldsymbol{a}_{\mathbf{0}}$ | $\boldsymbol{P}$ | $\boldsymbol{r} / \boldsymbol{a}_{\mathbf{0}}$ | $\boldsymbol{P}$ | $\boldsymbol{r} / \boldsymbol{a}_{\mathbf{0}}$ | $\boldsymbol{P}$ |
| :---: | :---: | :---: | :---: | ---: | :---: | :---: | :---: |
| 0.000 | 0.0000 | 0.280 | -0.0993 | 2.800 | -0.2524 | 13.000 | -0.0634 |
| 0.005 | 0.0142 | 0.300 | -0.0972 | 3.000 | -0.2926 | 14.000 | -0.0443 |
| 0.010 | 0.0257 | 0.350 | -0.0830 | 3.200 | -0.3279 | 15.000 | -0.0305 |
| 0.015 | 0.0349 | 0.400 | -0.0598 | 3.400 | -0.3583 | 16.000 | ,- 0209 |
| 0.020 | 0.0421 | 0.450 | -0.0312 | 3.600 | -0.3840 | 17.000 | -0.0138 |
| 0.030 | 0.0509 | 0.500 | -0.0003 | 3.800 | -0.4052 | 18.000 | -0.0095 |
| 0.040 | 0.0540 | 0.550 | 0.0307 | 4.000 | -0.4221 | 19.000 | -0.0063 |
| 0.050 | 0.0527 | 0.600 | 0.0601 | 4.500 | -0.4476 | 20.000 | -0.0042 |
| 0.060 | 0.0480 | 0.700 | 0.1105 | 5.000 | -0.4530 | 21.000 | -0.0028 |
| 0.070 | 0.0409 | 0.800 | 0.1465 | 5.500 | -0.4430 | 22.000 | -0.0018 |
| 0.080 | 0.0321 | 0.900 | 0.1679 | 6.000 | -0.4220 | 23.000 | -0.0012 |
| 0.090 | 0.0220 | 1.000 | 0.1761 | 6.500 | -0.3937 | 24.000 | -0.0008 |
| 0.100 | 0.0113 | 1.100 | 0.1734 | 7.000 | -0.3609 | 25.000 | -0.0005 |
| 0.120 | -0.0108 | 1.200 | 0.1623 | 7.500 | -0.3264 | 26.000 | -0.0003 |
| 0.140 | -0.0321 | 1.400 | 0.1226 | 8.000 | -0.2916 | 27.000 | -0.0002 |
| 0.160 | -0.0511 | 1.600 | 0.0699 | 8.500 | -0.2578 | 28.000 | -0.0001 |
| 0.180 | -0.0673 | 1.800 | 0.0119 | 9.000 | -0.2261 | 29.000 | -0.0001 |
| 0.200 | -0.0801 | 2.000 | -0.0470 | 9.500 | -0.1967 | 30.000 | 0.0000 |
| 0.220 | -0.0896 | 2.200 | -0.1040 | 10.000 | -0.1700 | 31.000 | 0.0000 |
| 0.240 | -0.0958 | 2.400 | -0.1578 | 11.000 | -0.1246 |  |  |
| 0.260 | -0.0989 | 2.600 | -0.2074 | 12.000 | -0.0896 |  |  |

Third, $\rho$ is set to the following top-hat distribution, which is a zeroth-order approximation to the actual distribution:

$$
\begin{equation*}
\rho(r)=\frac{3}{4 \pi R^{3}} \tag{52}
\end{equation*}
$$

for $r \leq R$ and $\rho=0$ otherwise. Repeating the derivation starting from Eq. 43 yields

$$
\begin{equation*}
\vec{B}_{n}=\frac{5 \mu_{0}}{16 \pi R^{3}} \vec{\mu}_{n} . \tag{53}
\end{equation*}
$$

In addition to the above torque-averaged fields, for comparison, the self-averaged fields from Eq. 38 are derived:

$$
\begin{equation*}
\vec{B}_{n}=\frac{2 \mu_{0}}{3} \rho(0) \vec{\mu}_{n} . \tag{54}
\end{equation*}
$$

For the Gaussian, tabulated, and top-hat distributions, $\rho(0)=\frac{8}{\pi^{3} R^{3}}, \frac{14.2}{\pi R^{3}}$, and $\frac{3}{4 \pi R^{3}}$, respectively; correspondingly, $\vec{B}_{n}=\frac{16 \mu_{0}}{3 \pi^{3} R^{3}} \vec{\mu}_{n}, \frac{28.4 \mu_{0}}{3 \pi R^{3}} \vec{\mu}_{n}$, and $\frac{\mu_{0}}{2 \pi R^{3}} \vec{\mu}_{n}$. These self-averaged fields, related to the Fermi contact interaction, are expected to be less compatible with the torque-based Landau-Lifshitz-Gilbert and CQD equations than the above torque-averaged fields.

The six solutions for $\vec{B}_{n}$ differ only by a constant factor. Eq. 53 , however, predicts the experimental observation (Fig. 4) most accurately, achieving a coefficient of determination of $R^{2}=0.9787$. The alternatives produce negative $R^{2}$, indicating worse accuracy than modeling with a horizontal line intercepting at the mean observation. Therefore, we choose the torqueaveraged field given by Eq. 53, rewritten below:

$$
\begin{equation*}
\vec{B}_{n}=\frac{5 \mu_{0}}{16 \pi R^{3}} \vec{\mu}_{n} . \tag{55}
\end{equation*}
$$

Reciprocally, the torque-averaged field from $\vec{\mu}_{e}$ applied on the nucleus is

$$
\begin{equation*}
\vec{B}_{e}=\frac{5 \mu_{0}}{16 \pi R^{3}} \vec{\mu}_{e} \tag{56}
\end{equation*}
$$

Appendix 2. Derivation of CQD equations of motion
We apply the Landau-Lifshitz-Gilbert equation to both $\hat{\mu}_{e}$ and $\hat{\mu}_{n}$, yielding

$$
\begin{equation*}
\frac{d \hat{\mu}_{e}}{d t}=\gamma_{e} \hat{\mu}_{e} \times\left(\vec{B}+\vec{B}_{n}\right)-k_{i} \hat{\mu}_{e} \times \frac{d \hat{\mu}_{e}}{d t} \tag{57}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d \widehat{\mu}_{n}}{d t}=\gamma_{n} \hat{\mu}_{n} \times\left(\vec{B}+\vec{B}_{e}\right)-k_{i} \hat{\mu}_{n} \times \frac{d \widehat{\mu}_{n}}{d t} . \tag{58}
\end{equation*}
$$

The external field is

$$
\vec{B}=\left(\begin{array}{c}
0  \tag{59}\\
B_{y} \\
B_{z}
\end{array}\right),
$$

where $B_{x}$ is neglected for brevity. The internal torque-averaged fields from the nucleus and the electron applied on each other (Appendix 1) are

$$
\begin{equation*}
\vec{B}_{n}=B_{n} \hat{\mu}_{n} \tag{60}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{B}_{e}=B_{e} \hat{\mu}_{e} \tag{61}
\end{equation*}
$$

We have

$$
\hat{\mu}_{e}=\left(\begin{array}{c}
\sin \theta_{e} \cos \phi_{e}  \tag{62}\\
\sin \theta_{e} \sin \phi_{e} \\
\cos \theta_{e}
\end{array}\right)
$$

and

$$
\hat{\mu}_{n}=\left(\begin{array}{c}
\sin \theta_{n} \cos \phi_{n}  \tag{63}\\
\sin \theta_{n} \sin \phi_{n} \\
\cos \theta_{n}
\end{array}\right)
$$

where $\theta$ and $\phi$ denote the polar and azimuthal angles, respectively.
Combining the above equations yields

$$
\begin{gather*}
\dot{\theta}_{e}=-\gamma_{e}\left[B_{y} \cos \phi_{e}+B_{n} \sin \theta_{n} \sin \left(\phi_{n}-\phi_{e}\right)\right]+k_{i} \dot{\phi}_{e} \sin \theta_{e},  \tag{64}\\
\dot{\theta}_{n}=-\gamma_{n}\left[B_{y} \cos \phi_{n}+B_{e} \sin \theta_{e} \sin \left(\phi_{e}-\phi_{n}\right)\right]+k_{i} \dot{\phi}_{n} \sin \theta_{n},  \tag{65}\\
\dot{\phi}_{e}=-\gamma_{e}\left\{B_{z}+B_{n} \cos \theta_{n}-\cot \theta_{e}\left[B_{y} \sin \phi_{e}+B_{n} \sin \theta_{n} \cos \left(\phi_{n}-\phi_{e}\right)\right]\right\}-\frac{k_{i} \dot{\theta}_{e}}{\sin \theta_{e}}, \tag{66}
\end{gather*}
$$

and

$$
\begin{equation*}
\dot{\phi}_{n}=-\gamma_{n}\left\{B_{z}+B_{e} \cos \theta_{e}-\cot \theta_{n}\left[B_{y} \sin \phi_{n}+B_{e} \sin \theta_{e} \cos \left(\phi_{e}-\phi_{n}\right)\right]\right\}-\frac{k_{i} \dot{\theta}_{n}}{\sin \theta_{n}} . \tag{67}
\end{equation*}
$$

To implement the second CQD postulate, we revise the sign of the last term in each of the above four equations, producing the CQD equations of motion (Eq. 5-8). Note that azimuthal angles are
not defined when the polar angles are 0 or $\pi$. Therefore, when $\theta_{e}=0$ or $\pi$, we use $\dot{\phi}_{e}=0$; when $\theta_{n}=0$ or $\pi$, we use $\dot{\phi}_{n}=0$.

If $B_{x} \neq 0$, the above equations can be extended by the following substitutions:

$$
\begin{aligned}
B_{y} \cos \phi_{e} & \rightarrow-B_{x} \sin \phi_{e}+B_{y} \cos \phi_{e} \\
B_{y} \cos \phi_{n} & \rightarrow-B_{x} \sin \phi_{n}+B_{y} \cos \phi_{n} \\
B_{y} \sin \phi_{e} & \rightarrow B_{x} \cos \phi_{e}+B_{y} \sin \phi_{e}
\end{aligned}
$$

and

$$
B_{y} \sin \phi_{n} \rightarrow B_{x} \cos \phi_{n}+B_{y} \sin \phi_{n} .
$$

Appendix 3. CQD derivation of density operator and wave function
CQD reproduces the quantum mechanical density operator and wave function with an isotropic angular distribution of co-quanta $\left(\hat{\mu}_{n}\right)$ and extends them with an anisotropic angular distribution of co-quanta.

For a given $\hat{\mu}_{e}$, the CQD prediction expressions (see Methods) for two independent realizations are written in dual spaces [17]:

$$
\begin{equation*}
\left|\hat{\mu}_{e} ® \hat{\mu}_{n i}\right\rangle=C_{i+}\left(\hat{\mu}_{e}, \hat{\mu}_{n i}\right)|+z\rangle+C_{i-}\left(\hat{\mu}_{e}, \hat{\mu}_{n i}\right) \exp \left(i \phi_{e}\right)|-z\rangle \tag{68}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\hat{\mu}_{e} \Subset \hat{\mu}_{n j}\right|=C_{j+}\left(\hat{\mu}_{e}, \hat{\mu}_{n j}\right)\langle+z|+C_{j-}\left(\hat{\mu}_{e}, \hat{\mu}_{n j}\right) \exp \left(-i \phi_{e}\right)\langle-z| \tag{69}
\end{equation*}
$$

Integer subscripts $i$ and $j$ denote independent realizations (i.e., $i \neq j$ ). Each binary coefficient represents either one or zero according to the branching condition (Eq. 10).

The pre-averaging density operator is defined as

$$
\begin{equation*}
\rho_{0} \stackrel{\text { def }}{=}\left|\hat{\mu}_{e} ® \hat{\mu}_{n i}\right\rangle\left\langle\hat{\mu}_{e} \Subset \hat{\mu}_{n j}\right|, \tag{70}
\end{equation*}
$$

which serves as a bridge to quantum mechanics [17]. Substitution of Eq. 68 and 69 results in

$$
\begin{equation*}
\rho_{0}=\left[C_{i+}|+z\rangle+C_{i-} \exp \left(i \phi_{e}\right)|-z\rangle\right]\left[C_{j+}\langle+z|+C_{j-} \exp \left(-i \phi_{e}\right)\langle-z|\right] . \tag{71}
\end{equation*}
$$

Expansion leads to

$$
\begin{gather*}
\rho_{0}=C_{i+} C_{j+}|+z\rangle\langle+z|+C_{i-} C_{j-}|-z\rangle\langle-z| \\
+C_{i+} C_{j-} \exp \left(-i \phi_{e}\right)|+z\rangle\langle-z|+C_{i-} C_{j+} \exp \left(+i \phi_{e}\right)|-z\rangle\langle+z| \tag{72}
\end{gather*}
$$

If the dual vectors represented identical realizations, the cross terms would vanish because the binary coefficients are mutually exclusive for each realization: $C_{i \pm} \cdot C_{i \mp}=0$ and $C_{j \pm} \cdot C_{j \mp}=$ 0 .

If $\hat{\mu}_{n}$ is random for a given $\hat{\mu}_{e}$, ensemble averaging $\rho_{0}$ over all realizations of $\hat{\mu}_{n}$, denoted by $\left\rangle_{n}\right.$, yields

$$
\begin{gather*}
\rho_{1}=\left\langle\rho_{0}\right\rangle_{n}=\left\langle C_{i+} C_{j+}\right\rangle_{n}|+z\rangle\langle+z|+\left\langle C_{i-} C_{j-}\right\rangle_{n}|-z\rangle\langle-z| \\
+\left\langle C_{i+} C_{j-} \exp \left(-i \phi_{e}\right)\right\rangle_{n}|+z\rangle\langle-z|+\left\langle C_{i-} C_{j+} \exp \left(+i \phi_{e}\right)\right\rangle_{n}|-z\rangle\langle+z| \tag{73}
\end{gather*}
$$

The following equations are invoked next: $\left\langle C_{i+}\right\rangle_{n}=\left\langle C_{j+}\right\rangle_{n}$, denoted by $\left\langle C_{+}\right\rangle_{n} ;\left\langle C_{i-}\right\rangle_{n}=$ $\left\langle C_{j-}\right\rangle_{n}$, denoted by $\left\langle C_{-}\right\rangle_{n}$. Given the independence of the two realizations $(i \neq j)$, we have $\left\langle C_{i \pm} C_{j \pm}\right\rangle_{n}=\left\langle C_{i \pm}\right\rangle_{n}\left\langle C_{j \pm}\right\rangle_{n}=\left\langle C_{ \pm}\right\rangle_{n}^{2}$ and $\left\langle C_{i \pm} C_{j \mp}\right\rangle_{n}=\left\langle C_{i \pm}\right\rangle_{n}\left\langle C_{j \mp}\right\rangle_{n}=\left\langle C_{+}\right\rangle_{n}\left\langle C_{-}\right\rangle_{n}$, yielding

$$
\begin{gather*}
\rho_{1}=\left\langle C_{+}\right\rangle_{n}^{2}|+z\rangle\langle+z|+\left\langle C_{-}\right\rangle_{n}^{2}|-z\rangle\langle-z| \\
+\left\langle C_{+}\right\rangle_{n}\left\langle C_{-}\right\rangle_{n} \exp \left(-i \phi_{e}\right)|+z\rangle\langle-z|+\left\langle C_{-}\right\rangle_{n}\left\langle C_{+}\right\rangle_{n} \exp \left(+i \phi_{e}\right)|-z\rangle\langle+z| \tag{74}
\end{gather*}
$$

Factorization yields

$$
\begin{equation*}
\rho_{1}=\left[\left\langle C_{+}\right\rangle_{n}|+z\rangle+\left\langle C_{-}\right\rangle_{n} \exp \left(i \phi_{e}\right)|-z\rangle\right]\left[\left\langle C_{+}\right\rangle_{n}\langle+z|+\left\langle C_{-}\right\rangle_{n} \exp \left(-i \phi_{e}\right)\langle-z|\right] \tag{75}
\end{equation*}
$$

For a pure state, invoking $\rho_{1}=\left|\hat{\mu}_{e}\right\rangle\left\langle\hat{\mu}_{e}\right|$ retrieves the following ket equation:

$$
\begin{equation*}
\left|\hat{\mu}_{e}\right\rangle=\left\langle C_{+}\right\rangle_{n}|+z\rangle+\left\langle C_{-}\right\rangle_{n} \exp \left(i \phi_{e}\right)|-z\rangle . \tag{76}
\end{equation*}
$$

If $\hat{\mu}_{n}$ follows the isotropic $p_{n 0}$ (Eq. 15), the expected probabilities of collapse are computed from Eq. 74 as follows:

$$
\begin{equation*}
\langle+z| \rho_{1}|+z\rangle=\left\langle C_{+}\right\rangle_{n}^{2}=\int_{\theta_{e}}^{\pi} p_{n 0} 2 \pi \sin \theta_{n} d \theta_{n}=\cos ^{2} \frac{\theta_{e}}{2} \tag{77}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle-z| \rho_{1}|-z\rangle=\left\langle C_{-}\right\rangle_{n}^{2}=\int_{0}^{\theta_{e}} p_{n 0} 2 \pi \sin \theta_{n} d \theta_{n}=\sin ^{2} \frac{\theta_{e}}{2} . \tag{78}
\end{equation*}
$$

The integration limits are based on the branching condition (Eq. 10). Because $p_{n 0}$ is isotropic (i.e., spherical), the two probabilities are proportional to the solid angles formed by the down and up sides of the cone shaped by the initial Bloch vector ${ }^{15}\left(\hat{\mu}_{e}\right)$ precessing over one cycle (Fig. S1). Each solid angle determines the probability of having the co-quantum on the corresponding side of the cone. In other words, the above two equations represent the probabilities of having the coquantum on the corresponding side of the cone.


Fig. S1. Illustration of the cone of $\hat{\mu}_{e}$ formed by precession around the external main field, $B_{0}$, over the first Larmor cycle (i.e., before collapse). $e$ : electron magnetic moment (principal quantum), $\hat{\mu}_{e}$. Any nuclear magnetic moment (co-quantum), $\hat{\mu}_{n}$, precessing around $B_{0}$ within the cone causes $\hat{\mu}_{e}$ to collapse down, whereas $\hat{\mu}_{n}$ precessing outside the cone causes $\hat{\mu}_{e}$ to collapse up. For a given polar angle $\theta_{e}$ of $\hat{\mu}_{e}$, the probability for the atom from the oven to reach the down branch in the first-stage Stern-Gerlach experiment is proportional to the solid angle of the cone because $\hat{\mu}_{n}$ follows an isotropic angular distribution. However, in subsequent-stage Stern-Gerlach experiments, the relation is revised because $\hat{\mu}_{n}$ follows an anisotropic angular distribution, such as the heart shape (Fig. 3, Inset a, solid line; Eq. 24).

Consequently, we reach the familiar quantum mechanical density operator,

$$
\begin{gather*}
\rho_{1}=\cos ^{2} \frac{\theta_{e}}{2}|+z\rangle\langle+z|+\sin ^{2} \frac{\theta_{e}}{2}|-z\rangle\langle-z| \\
+\cos \frac{\theta_{e}}{2} \sin \frac{\theta_{e}}{2} \exp \left(-i \phi_{e}\right)|+z\rangle\langle-z|+\sin \frac{\theta_{e}}{2} \cos \frac{\theta_{e}}{2} \exp \left(+i \phi_{e}\right)|-z\rangle\langle+z| \tag{79}
\end{gather*}
$$

Factorization of the density operator yields

$$
\begin{equation*}
\rho_{1}=\left[\cos \frac{\theta_{e}}{2}|+z\rangle+\sin \frac{\theta_{e}}{2} \exp \left(i \phi_{e}\right)|-z\rangle\right]\left[\cos \frac{\theta_{e}}{2}\langle+z|+\sin \frac{\theta_{e}}{2} \exp \left(-i \phi_{e}\right)\langle-z|\right] . \tag{80}
\end{equation*}
$$

Invoking $\rho_{1}=\left|\hat{\mu}_{e}\right\rangle\left\langle\hat{\mu}_{e}\right|$ for a pure state retrieves the following familiar quantum mechanical ket equation:

$$
\begin{equation*}
\left|\hat{\mu}_{e}\right\rangle=\cos \frac{\theta_{e}}{2}|+z\rangle+\sin \frac{\theta_{e}}{2} \exp \left(i \phi_{e}\right)|-z\rangle . \tag{81}
\end{equation*}
$$

Therefore, CQD statistically reproduces the quantum mechanical wave function along with the probability amplitudes. As shown by Eq. 76, the moduli of probability amplitudes originate from averaging the binary coefficients in the CQD prediction expression.

If $\hat{\mu}_{e}$ is also random, further ensemble averaging $\rho_{1}$ over all realizations, denoted by $\left\rangle_{e}\right.$, yields

$$
\begin{gather*}
\rho_{2}=\left\langle\rho_{1}\right\rangle_{e}=\left\langle\cos ^{2} \frac{\theta_{e}}{2}\right\rangle_{e}|+z\rangle\langle+z|+\left\langle\sin ^{2} \frac{\theta_{e}}{2}\right\rangle_{e}|-z\rangle\langle-z| \\
+\left\langle\cos \frac{\theta_{e}}{2} \sin \frac{\theta_{e}}{2} \exp \left(-i \phi_{e}\right)\right\rangle_{e}|+z\rangle\langle-z|+\left\langle\sin \frac{\theta_{e}}{2} \cos \frac{\theta_{e}}{2} \exp \left(+i \phi_{e}\right)\right\rangle_{e}|-z\rangle\langle+z| \tag{82}
\end{gather*}
$$

If $\hat{\mu}_{e}$ follows the isotropic $p_{e 0}$ (Eq. 19), the probabilities of collapse are computed as follows:

$$
\begin{equation*}
\langle+z| \rho_{2}|+z\rangle=\left\langle\cos ^{2} \frac{\theta_{e}}{2}\right\rangle_{e}=\int_{0}^{\pi}\left[\cos ^{2} \frac{\theta_{e}}{2}\right] p_{e 0} 2 \pi \sin \theta_{e} d \theta_{e}=\frac{1}{2} \tag{83}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle-z| \rho_{2}|-z\rangle=\left\langle\sin ^{2} \frac{\theta_{e}}{2}\right\rangle_{e}=\int_{0}^{\pi}\left[\sin ^{2} \frac{\theta_{e}}{2}\right] p_{e 0} 2 \pi \sin \theta_{e} d \theta_{e}=\frac{1}{2} . \tag{84}
\end{equation*}
$$

The cross terms vanish due to the azimuthal integration of $\exp \left(-i \phi_{e}\right)$ over a full cycle of $\hat{\mu}_{e}$ :

$$
\begin{gather*}
\langle+z| \rho_{2}|-z\rangle=\left\langle\cos \frac{\theta_{e}}{2} \sin \frac{\theta_{e}}{2} \exp \left(-i \phi_{e}\right)\right\rangle_{e} \\
=\int_{0}^{\pi} \cos \frac{\theta_{e}}{2} \sin \frac{\theta_{e}}{2}\left[\int_{0}^{2 \pi} \exp \left(-i \phi_{e}\right) p_{e 0} d \phi_{e}\right] \sin \theta_{e} d \theta_{e}=0 \tag{85}
\end{gather*}
$$

and

$$
\begin{gather*}
\langle+z| \rho_{2}|-z\rangle=\left\langle\sin \frac{\theta_{e}}{2} \cos \frac{\theta_{e}}{2} \exp \left(+i \phi_{e}\right)\right\rangle_{e} \\
=\int_{0}^{\pi} \sin \frac{\theta_{e}}{2} \cos \frac{\theta_{e}}{2}\left[\int_{0}^{2 \pi} \exp \left(+i \phi_{e}\right) p_{e 0} d \phi_{e}\right] \sin \theta_{e} d \theta_{e}=0 . \tag{86}
\end{gather*}
$$

Therefore, we reach

$$
\begin{equation*}
\rho_{2}=\frac{1}{2}|+z\rangle\langle+z|+\frac{1}{2}|-z\rangle\langle-z| . \tag{87}
\end{equation*}
$$

This familiar quantum mechanical density operator represents the mixed state and cannot be factorized into a product of two pure-state wave functions.

If $\hat{\mu}_{n}$ follows the heart-shaped $p_{n 1}$ (Eq. 24), the probabilities of collapse become

$$
\begin{equation*}
\langle+z| \rho_{1}|+z\rangle=\left\langle C_{+}\right\rangle_{n}^{2}=\int_{\theta_{e}}^{\pi} p_{n 1} 2 \pi \sin \theta_{n} d \theta_{n}=1-\sin ^{4} \frac{\theta_{e}}{2} \tag{88}
\end{equation*}
$$

instead of Eq. 77 and

$$
\begin{equation*}
\langle-z| \rho_{1}|-z\rangle=\left\langle C_{-}\right\rangle_{n}^{2}=\int_{0}^{\theta_{e}} p_{n 1} 2 \pi \sin \theta_{n} d \theta_{n}=\sin ^{4} \frac{\theta_{e}}{2} \tag{89}
\end{equation*}
$$

instead of Eq. 78. Because $p_{n 1}$ is anisotropic (i.e., heart-shaped), the two probabilities are no longer simply proportional to the solid angles. However, the above two equations still represent the probabilities of having the co-quantum on the corresponding side of the cone (Fig. S1).

Accordingly, the wave function becomes

$$
\begin{equation*}
\left|\hat{\mu}_{e}\right\rangle=\sqrt{1-\sin ^{4} \frac{\theta_{e}}{2}}|+z\rangle+\sin ^{2} \frac{\theta_{e}}{2} \exp \left(i \phi_{e}\right)|-z\rangle \tag{90}
\end{equation*}
$$

instead of Eq. 81.

## Appendix 4. Derivation of Majorana formula

The Majorana formula [10] or its Landau-Zener variant [12-14] was derived most intuitively in 2005 by Wittig [56]. Recent rederivations of the Majorana formula can be found in Wilczek [57] and Kofman [58]. Majorana stated that both the classical and the quantum-mechanical treatments require integration of the same differential equations [10, 11]. For completeness here, we follow Majorana's variable transformations and then abridge Wittig's solution but with a slightly altered contour integration.

For the inner rotation chamber (Fig. 3, IR), the B field along the $y$ axis is approximated using a magnetic quadrupole (Fig. S2) [10, 11]:


Fig. S2. Illustration of the magnetic field versus the $y$ location of the atom in the inner rotation chamber (i.e., the middle stage). $y$ denotes the axis of the atomic beam. Here, $B_{y}>0$, and $G=$ $\partial B_{z} / \partial y>0$. For a given current, $B_{y}$ is invariant with $y$ while $B_{z}$ is proportional to $y$ (i.e., negative for $y<0$, zero at $y=0$, and positive for $y>0$ ). At $y= \pm z_{a}$, we have $B_{z}= \pm B_{y}$ or $\left|B_{z}\right|=B_{y}$.

$$
\begin{gather*}
B_{x}=0,  \tag{91}\\
B_{y}=G z_{a} \tag{92}
\end{gather*}
$$

and

$$
\begin{equation*}
B_{z}=G v t . \tag{93}
\end{equation*}
$$

Here, $G$ is the derivative of $B_{z}$ with respect to $y$ (i.e., the gradient magnitude of $B_{z}, \partial B_{z} / \partial y$ ), $z_{a}$ $\left(1.05 \times 10^{-4} \mathrm{~m}\right)$ is the vertical distance of the atomic beam from the center of the wire, $v(800 \mathrm{~m}$ $\mathrm{s}^{-1}$ ) is the most likely speed of atoms, and $t$ is time set to zero at the null point of $B_{z} . G$ is given by

$$
\begin{equation*}
G=\frac{\partial B_{z}}{\partial y}=\frac{2 \pi}{\mu_{0} I} B_{r}^{2} . \tag{94}
\end{equation*}
$$

Here, $I$ denotes the current carried by the wire along the $-x$ axis, and $B_{r}\left(0.42 \times 10^{-4} \mathrm{~T}\right)$ denotes the uniformly distributed remnant (residual) fringe magnetic flux density, which is parallel with the $+z$ axis. The magnetic field generated by the wire cancels the remnant field at the null point $(\mathrm{NP})$ to produce an approximate quadrupole (Fig. 3, Inset b). For a given current, $B_{y}$ is constant while $B_{z}$ varies linearly with distance from the point where $B_{z}=0$.

Majorana neglected the nuclear magnetic moment and induction. The Bloch equation leads to

$$
\begin{equation*}
\dot{\theta}_{e}=-\gamma_{e} B_{y} \cos \phi_{e} \tag{95}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{\phi}_{e}=-\gamma_{e}\left[B_{z}-\cot \theta_{e} B_{y} \sin \phi_{e}\right], \tag{96}
\end{equation*}
$$

which agree with Eq. 5 and 7 for $B_{n}=0$ and $k_{i}=0$.
Majorana transformed the polar and azimuthal angles into probability amplitudes then solved the transformed equations [10, 11]. We let

$$
\begin{equation*}
|\hat{\mu}\rangle=\binom{c_{1}}{c_{2}} \tag{97}
\end{equation*}
$$

The Schrödinger equation becomes

$$
\begin{equation*}
i \hbar \frac{d}{d t}\binom{c_{1}}{c_{2}}=H\binom{c_{1}}{c_{2}} \tag{98}
\end{equation*}
$$

The Hamiltonian is

$$
\begin{equation*}
H=-\frac{1}{2} \hbar \gamma_{e} \vec{B} \cdot \vec{\sigma} . \tag{99}
\end{equation*}
$$

Substituting the Pauli matrices $\vec{\sigma}$ yields

$$
H=-\frac{1}{2} \hbar \gamma_{e}\left[B_{x}\left(\begin{array}{ll}
0 & 1  \tag{100}\\
1 & 0
\end{array}\right)+B_{y}\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right)+B_{z}\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)\right] .
$$

Merging terms gives

$$
H=-\frac{1}{2} \hbar \gamma_{e}\left(\begin{array}{cc}
B_{z} & B_{x}-i B_{y}  \tag{101}\\
B_{x}+i B_{y} & -B_{z}
\end{array}\right)
$$

Majorana defined the following dimensionless variables for time and adiabaticity, respectively $[10,11]$ :

$$
\begin{equation*}
\tau=\frac{1}{2} \sqrt{\left|\gamma_{e} G v\right|} \cdot t \tag{102}
\end{equation*}
$$

and

$$
\begin{equation*}
k_{m}=\frac{\left|\gamma_{e}\right| B_{y}}{G v / B_{y}}=\frac{\left|\gamma_{e}\right| B_{y}^{2}}{G v}=\frac{z_{a}}{v}\left|\gamma_{e}\right| B_{y} . \tag{103}
\end{equation*}
$$

The numerator, $\left|\gamma_{e}\right| B_{y}$, in the first fraction above represents the Larmor frequency about the $y$ axis, whereas the denominator, $G v / B_{y}$, represents approximately the rotation frequency of the field.

Accordingly, the Schrödinger equation is simplified to

$$
\begin{equation*}
\frac{d}{d \tau}\binom{c_{1}}{c_{2}}=-i\binom{2 \tau c_{1}-i \sqrt{k_{m}} c_{2}}{i \sqrt{k_{m}} c_{1}-2 \tau c_{2}} \tag{104}
\end{equation*}
$$

Majorana defined the following transformation of variables [10, 11], which is analogous to heterodyne detection with a chirped local oscillator to remove high-frequency signals:

$$
\begin{equation*}
\binom{c_{1}}{c_{2}}=\binom{\exp \left(-i \tau^{2}\right) f}{\exp \left(+i \tau^{2}\right) g} \tag{105}
\end{equation*}
$$

The Schrödinger equation becomes

$$
\begin{equation*}
\frac{d}{d \tau}\binom{f}{g}=\sqrt{k_{m}}\binom{-\exp \left(+2 i \tau^{2}\right) g}{+\exp \left(-2 i \tau^{2}\right) f} \tag{106}
\end{equation*}
$$

Eliminating variables yields

$$
\begin{equation*}
\left(\frac{d^{2}}{d \tau^{2}} \mp 4 i \tau \frac{d}{d \tau}+k_{m}\right)\binom{f}{g}=0 \tag{107}
\end{equation*}
$$

For $|f(-\infty)|=1$, we solve for $|f(+\infty)|$. Following Wittig [56], we rewrite the equation for $f$ as

$$
\begin{equation*}
4 i \frac{d f}{f}=\frac{d \tau}{\tau}\left(\frac{d^{2} f}{d \tau^{2}} \frac{1}{f}+k_{m}\right) . \tag{108}
\end{equation*}
$$

Integrating over the entire flight yields

$$
\begin{equation*}
4 i \int_{-\infty}^{+\infty} \frac{d f}{f}=\int_{-\infty}^{+\infty} \frac{d \tau}{\tau}\left(\frac{d^{2} f}{d \tau^{2}} \frac{1}{f}+k_{m}\right) \tag{109}
\end{equation*}
$$

We select a positively oriented and indented contour that excludes the singularity at $\tau=0$ in the complex plane (Fig. S3), whereas the opposite orientation would yield an unphysical outcome. Consequently, we have

$$
\begin{equation*}
4 i \ln \frac{f(+\infty)}{f(-\infty)}=\oint \quad-\lim _{\varepsilon \rightarrow 0} \int_{\operatorname{arc}} \quad-\lim _{R \rightarrow \infty} \int_{\operatorname{Arc}} \frac{d \tau}{\tau}\left(\frac{d^{2} f}{d \tau^{2}} \frac{1}{f}+k_{m}\right) \tag{110}
\end{equation*}
$$

Here, $\varepsilon$ is the radius of the small indenting semicircular arc, $R$ is the radius of the large semicircular arc, and $\tau$ is now made complex without substitution to a new complex variable (because the typically adopted $z$ is used already for space). Because no pole is inside the contour, the first integral on the right side vanishes.

At $\tau=0$, Eq. 107 gives the residue,

$$
\begin{equation*}
\operatorname{Res}_{0}=\left.\left(\frac{d^{2} f}{d \tau^{2}} \frac{1}{f}+k_{m}\right)\right|_{\tau=0}=0 \tag{111}
\end{equation*}
$$

Thus, the second integral along the small arc on the right side of Eq. 110 vanishes too.
As $\tau \rightarrow \infty, \frac{d^{2} f}{d \tau^{2}} \frac{1}{f} \rightarrow 0$ [56]; substitution into the third integral along the large arc yields

$$
\begin{equation*}
4 i \ln \frac{f(+\infty)}{f(-\infty)}=-k_{m} \lim _{R \rightarrow \infty} \int_{\operatorname{Arc}} \frac{d \tau}{\tau} . \tag{112}
\end{equation*}
$$

Letting $\tau=R \exp (i \beta)$ leads to

$$
\begin{equation*}
\ln \frac{f(+\infty)}{f(-\infty)}=-\frac{k_{m}}{4 i} \lim _{R \rightarrow \infty} \int_{0}^{\pi} \frac{R \exp (i \beta) i d \beta}{R \exp (i \beta)}=-\frac{\pi k_{m}}{4} . \tag{113}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\frac{|f(+\infty)|}{|f(-\infty)|}=\exp \left(-\frac{\pi k_{m}}{4}\right) . \tag{114}
\end{equation*}
$$

Substituting $|f(-\infty)|=1$, we reach

$$
\begin{equation*}
|f(+\infty)|=\exp \left(-\frac{\pi k_{m}}{4}\right) \tag{115}
\end{equation*}
$$



Fig. S3. Contour on the complex plane of $\tau$ for the integration. The solid circle indicates the pole at $\tau=0$. Adapted from Wittig [56].

Majorana $[10,11]$ reasoned that because $B_{z}$ reverses its orientation along the flight path, the probability of spin flip, $W_{m}$, is given by $|f(+\infty)|^{2}$ instead of $|g(+\infty)|^{2}$. The further justification that we found is the initial adiabatic flip induced when the atom passes above the wire [47]. Therefore, we obtain

$$
\begin{equation*}
W_{m}=\exp \left(-\frac{\pi k_{m}}{2}\right) \tag{116}
\end{equation*}
$$

Here, $k_{m}>0$. If $k_{m}<0$, one may extend the solution to $\exp \left(-\frac{\pi\left|k_{m}\right|}{2}\right)$. Substitution of Eq. 103 results in

$$
\begin{equation*}
W_{m}=\exp \left(-\frac{\pi z_{a}}{2 v}\left|\gamma_{e}\right| B_{y}\right) \tag{117}
\end{equation*}
$$

## Appendix 5. Derivation of CQD formula

We derive the CQD formula for the probability of spin flip in the inner rotation chamber (Fig. 3, IR) in the presence of both the quadrupole field and the nuclear magnetic moment.

The average polar angle of $\vec{B}_{n},\left\langle\theta_{n}\right\rangle$, is derived from the heart shape given by Eq. 24:

$$
\begin{equation*}
\left\langle\theta_{n}\right\rangle=\int_{0}^{\pi} \theta_{n} p_{n 1} 2 \pi \sin \theta_{n} d \theta_{n}=5 \pi / 8 \tag{118}
\end{equation*}
$$

To reach an approximate analytical solution, we hold $\theta_{n}$ at $\left\langle\theta_{n}\right\rangle$ as a representative value throughout the inner rotation chamber.

The presence of $\vec{B}_{n}$ alters both the remnant field due to the projection of $\vec{B}_{n}$ to the $z$ axis, given by $B_{n} \cos \left\langle\theta_{n}\right\rangle$, and the transverse field due to the transverse projection of $\vec{B}_{n}$, represented by $B_{n} \sin \left\langle\theta_{n}\right\rangle \exp \left(i \phi_{n}\right)$. To extend the Majorana solution presented in Appendix 4, we first substitute the remnant field as follows:

$$
\begin{equation*}
B_{r} \rightarrow B_{r}^{\prime}=B_{r}+B_{n} \cos \left\langle\theta_{n}\right\rangle \tag{119}
\end{equation*}
$$

Accordingly, we update the field gradient:

$$
\begin{equation*}
G=\frac{2 \pi}{\mu_{0} I} B_{r}^{2} \rightarrow G^{\prime}=\frac{2 \pi}{\mu_{0} I} B_{r}^{\prime 2} . \tag{120}
\end{equation*}
$$

The quadrupole field along the $y$ axis is still given by the same equations but with the corrected field gradient:[10, 11]

$$
\begin{gather*}
B_{x}^{\prime}=0  \tag{121}\\
B_{y}^{\prime}=G^{\prime} z_{a} \tag{122}
\end{gather*}
$$

and

$$
\begin{equation*}
B_{z}^{\prime}=G^{\prime} v t \tag{123}
\end{equation*}
$$

Following Majorana [10, 11], we define the dimensionless time as

$$
\begin{equation*}
\tau=\frac{1}{2} \sqrt{\left|\gamma_{e} G^{\prime} v\right|} \cdot t \tag{124}
\end{equation*}
$$

and the dimensionless adiabaticity parameter due to $B_{y}^{\prime}$ as

$$
\begin{equation*}
k_{0}=\frac{\left|\gamma_{e}\right| B_{y}^{\prime}}{G^{\prime} v / B_{y}^{\prime}}=\frac{z_{a}}{v}\left|\gamma_{e}\right| B_{y}^{\prime} . \tag{125}
\end{equation*}
$$

The numerator, $\left|\gamma_{e}\right| B_{y}^{\prime}$, in the first fraction above represents the Larmor frequency about the $y$ axis, whereas the denominator, $G^{\prime} v / B_{y}^{\prime}$, represents approximately the rotation frequency of the field. We similarly define the adiabaticity parameter due to the transverse field $B_{n} \sin \left\langle\theta_{n}\right\rangle$ as

$$
\begin{equation*}
k_{1}=\frac{\left|\gamma_{e}\right|\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)}{G^{\prime} v /\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)}=\frac{z_{a}}{v}\left|\gamma_{e}\right| \frac{\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)^{2}}{B_{y}^{\prime}} . \tag{126}
\end{equation*}
$$

We also compute the dimensionless counterpart of the Larmor frequency $\omega_{n}$ as

$$
\begin{equation*}
w_{n}=\frac{d \phi_{n}}{d \tau}=\frac{d \phi_{n}}{d t} \frac{d t}{d \tau}=\frac{2}{\sqrt{\left|\gamma_{e} G^{\prime} v\right|}} \omega_{n} \tag{127}
\end{equation*}
$$

where $\omega_{n}=d \phi_{n} / d t$ was invoked.
Then, we substitute the total transverse field for the Schrödinger equation:

$$
\begin{equation*}
B_{x} \rightarrow B_{x}^{\prime}+B_{n} \sin \left\langle\theta_{n}\right\rangle \cos \phi_{n} \tag{128}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{y} \rightarrow B_{y}^{\prime}+B_{n} \sin \left\langle\theta_{n}\right\rangle \sin \phi_{n} \tag{129}
\end{equation*}
$$

Following the procedure presented in Appendix 4, we revise the Schrödinger equation to

$$
\frac{d}{d \tau}\binom{c_{1}}{c_{2}}=-i\left(\begin{array}{cc}
2 \tau & \sqrt{k_{1}} \exp \left(-i \phi_{n}\right)-i \sqrt{k_{0}}  \tag{130}\\
\sqrt{k_{1}} \exp \left(i \phi_{n}\right)+i \sqrt{k_{0}} & -2 \tau
\end{array}\right)\binom{c_{1}}{c_{2}}
$$

Defining

$$
\begin{equation*}
\binom{c_{1}}{c_{2}}=\binom{\exp \left(-i \tau^{2}\right) f(\tau)}{\exp \left(+i \tau^{2}\right) g(\tau)} \tag{131}
\end{equation*}
$$

we reach

$$
\begin{equation*}
\frac{d}{d \tau}\binom{f(\tau)}{g(\tau)}=\binom{\left[-i \sqrt{k_{1}} \exp \left(-i \phi_{n}\right)-\sqrt{k_{0}}\right] \exp \left(+2 i \tau^{2}\right) g(\tau)}{\left[-i \sqrt{k_{1}} \exp i \phi_{n}+\sqrt{k_{0}}\right] \exp \left(-2 i \tau^{2}\right) f(\tau)} . \tag{132}
\end{equation*}
$$

Eliminating $g$ yields

$$
\begin{equation*}
4 i \frac{d f}{f}=\frac{d \tau}{\tau-\frac{\sqrt{k_{1}} w_{n}}{4\left[\sqrt{k_{1}}-i \sqrt{k_{0}} \exp \left(i \phi_{n}\right)\right]}}\left\{\frac{d^{2} f \tau^{2}}{d \tau^{2}}+\left[k_{0}+k_{1}+2 \sqrt{k_{0} k_{1}} \sin \left(\phi_{n}\right)\right]\right\} . \tag{133}
\end{equation*}
$$

In the limiting case that $k_{0} \gg k_{1}$, i.e., the extremely low-current region, we set $k_{1}=0$, reducing Eq. 133 to

$$
\begin{equation*}
4 i \frac{d f}{f}=\frac{d \tau}{\tau}\left\{\frac{d^{2} f}{d \tau^{2}} \frac{1}{f}+k_{0}\right\}, \tag{134}
\end{equation*}
$$

which is the same as Eq. 108. The solution with $|f(-\infty)|=1$ is given by Eq. 115:

$$
\begin{equation*}
|f(+\infty)|=\exp \left(-\frac{\pi k_{0}}{4}\right) \tag{135}
\end{equation*}
$$

From Eq. 25 in Results, the fraction of flip is given by $|f(+\infty)|^{4}$ instead of $|f(+\infty)|^{2}$ due to the heart-shaped $p_{n 1}$ :

$$
\begin{equation*}
W_{2}=\exp \left(-\pi k_{0}\right)=\exp \left(-\pi \frac{z_{a}}{v}\left|\gamma_{e}\right| B_{y}^{\prime}\right) . \tag{136}
\end{equation*}
$$

$W_{2}$ is shown in Fig. S4 below and Fig. 4. Because the pole in Eq. 134 is at $\tau=0$, we call this effect null-point rotation.

Conversely, if we set $k_{0}=0$, Eq. 133 reduces to

$$
\begin{equation*}
4 i \frac{d f}{f}=\frac{d \tau}{\tau-\frac{w_{n}}{4}}\left\{\frac{d^{2} f}{d \tau^{2}} \frac{1}{f}+k_{1}\right\}, \tag{137}
\end{equation*}
$$

which resembles Eq. 108, however, with the pole shifted from 0 to $w_{n} / 4$. The solution is likewise obtained as

$$
\begin{equation*}
|f(+\infty)|=\exp \left(-\frac{\pi k_{1}}{4}\right) \tag{138}
\end{equation*}
$$

Similarly, the fraction of flip is

$$
\begin{equation*}
W_{R}=\exp \left(-\pi k_{1}\right) \tag{139}
\end{equation*}
$$

which is shown in Fig. S4. The pole $\tau=w_{n} / 4$ is converted using Eq. 124 and Eq. 127 to dimensional quantities as $\omega_{e z}(t)=\omega_{n}(t)$, where $\omega_{e z}$ denotes the Larmor frequency of $\vec{\mu}_{e}$ about the $z$ axis and $\omega_{n}$ denotes the Larmor frequency of $\vec{\mu}_{n}$. Therefore, the flip is due to precession resonance between the magnetic moments of the nucleus and the electron, which we refer to as nuclear-resonant rotation.


Fig. S4. Fraction of spin flip versus wire current. As the current increases from 0.01 A to 0.5 A , $k_{0}$ decreases inversely proportionally with the current from 1.701 to 0.034 , and $k_{1}$ increases proportionally with the current from 0.038 to 1.891 . When $B_{y}^{\prime}=B_{n} \sin \left\langle\theta_{n}\right\rangle=0.11 \times 10^{-4} \mathrm{~T}$, $k_{0}=k_{1}$; the corresponding current equals 0.067 A , about which the dashed and dash-dotted curves are mirror symmetric on the semilog plot. Experiment: Frisch-Segrè experiment.

There are three terms inside the square brackets in Eq. 133: $k_{0}+k_{1}+2 \sqrt{k_{0} k_{1}} \sin \left(\phi_{n}\right)$. A direct combination of the first two terms yields the solid-line curve in Fig. S4, which predicts the fraction of flip accurately at the two ends but only qualitatively in the intermediate region. Below, we combine the terms more quantitatively. For an ensemble of atoms, the nuclear magnetic moment of each atom is given a random initial phase, $\phi_{n 0}$, at $t=0$.

Given Eq. 25 in Results, we extend the Majorana solution (see Appendix 4) to

$$
\begin{equation*}
W_{\mathrm{cqd}}=\sin ^{4}\left(\frac{\alpha_{r}}{2}\right)=\exp \left(-E_{r 0}-E_{r 1}-E_{i}\right), \tag{140}
\end{equation*}
$$

where $\alpha_{r}$ represents the polar rotation by the inner rotation chamber from the initial $\theta_{e}=0$ and $E_{r 0}, E_{r 1}, E_{i}$ represent contributions from null-point rotation, nuclear-resonant rotation, and induction, respectively. The fourth power is due to the heart-shaped $p_{n 1}$.

The null-point rotation exponent is determined by the quadrature-summed (or the root-mean-squared) $B$ field on the $x y$ plane:

$$
\begin{equation*}
E_{r 0}=\frac{\pi z_{a}}{v}\left|\gamma_{e}\right| \sqrt{B_{y}^{\prime 2}+\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)^{2}} . \tag{141}
\end{equation*}
$$

The quadrature sum can be derived through $\left.\langle | i B_{y}^{\prime}+\left.B_{n} \sin \left\langle\theta_{n}\right\rangle \exp \left(i \phi_{n 0}\right)\right|^{2}\right\rangle^{1 / 2}$, where the ensemble average is over a uniform distribution of $\phi_{n 0}$ and $B_{n} \sin \left\langle\theta_{n}\right\rangle \exp \left(i \phi_{n 0}\right)$ is the transverse component of $\vec{B}_{n}$ at the null point. One may consider $\pi B_{y}^{\prime} / G^{\prime}$, yielding $\pi z_{a}$ (Eq. 122), as the effective flight path-length for the null-point rotation, $\pi y_{r 0}$ (Fig. S5). Here, $\pi y_{r 0}=\pi z_{a}=$ $\pi \times 0.105 \times 10^{-3}=0.33 \times 10^{-3} \mathrm{~m}$, which is a constant independent of the wire current. If $B_{n}=$ 0 , the quadrature sum reduces to $B_{y}^{\prime}$ as expected. $E_{r 0}$ is responsible for Curve 3 in Fig. 4, which predicts the experimental observation in the low-current region accurately. Further description can be found above Eq. 28.


Fig. S5. Illustration of the magnetic field versus the $y$ location of the atom in the inner rotation chamber (i.e., the middle stage). Here, $y_{r 1} / y_{r 0}=3$, for the current of 0.2 A . For a given current, $B_{y}^{\prime}$ is invariant with $y$ while $B_{z}^{\prime}$ is proportional to $y$ (i.e., negative for $y<0$, zero at $y=0$, and positive for $y>0) . B_{n} \sin \left\langle\theta_{n}\right\rangle$ is the transverse projection of $\vec{B}_{n}$ on the $x y$ plane. $G^{\prime}=\partial B_{z}^{\prime} / \partial y$.

At high currents where $I \geq 0.067 \mathrm{~A}, B_{y}^{\prime}$ becomes less than $B_{n} \sin \left\langle\theta_{n}\right\rangle$, and $k_{1} \geq k_{0}$; consequently, nuclear-resonant rotation due to the rotating transverse component of $\vec{B}_{n}$ becomes substantial. The resonant-rotation exponent is approximated heuristically as

$$
\begin{equation*}
E_{r 1}=\frac{1}{2}\left[\frac{\pi y_{r 1}}{v}\left|\gamma_{e}\right| B_{n} \sin \left\langle\theta_{n}\right\rangle\right]^{2}\left\{\frac{\pi y_{r 1}}{v} \frac{1}{T_{n}}\right\} . \tag{142}
\end{equation*}
$$

The term in the square bracket is analogous to the right-hand side of Eq. 141. This estimation is inspired by the following approximation for small fluctuations of a random variable $X$ :

$$
\begin{equation*}
\langle\exp (X-\bar{X})\rangle \approx \exp \left(\frac{1}{2}\left\langle(X-\bar{X})^{2}\right\rangle\right)=\exp \left(\frac{1}{2} \operatorname{Var}[X]\right) \tag{143}
\end{equation*}
$$

where Var denotes variance.
In analogy to $\pi y_{r 0}=\pi B_{y}^{\prime} / G^{\prime}$ defined below Eq. 141, the effective flight path-length for nuclear-resonant rotation, $\pi y_{r 1}$, is defined as (Fig. S5)

$$
\begin{equation*}
\pi y_{r 1}=\frac{\pi B_{n} \sin \left\langle\theta_{n}\right\rangle}{G^{\prime}}=\frac{\pi z_{a} B_{n} \sin \left\langle\theta_{n}\right\rangle}{B_{y}^{\prime}} \tag{144}
\end{equation*}
$$

Because $B_{e}$ is far greater than the field in the inner rotation chamber, the Larmor period of the nuclear magnetic moment is approximated to be

$$
\begin{equation*}
T_{n}=\frac{2 \pi}{\gamma_{n} B_{e}} . \tag{145}
\end{equation*}
$$

The curly bracket term in Eq. 142 represents the fraction of the Larmor period of the nuclear magnetic moment, denoted by $f_{r 1}$, precessed during the flight time over $\pi y_{r 1}$ :

$$
\begin{equation*}
f_{r 1}=\frac{\pi y_{r 1}}{v T_{n}} \tag{146}
\end{equation*}
$$

One may consider $\pi y_{r 1} / v$ as the effective nuclear-resonant time. Note that $f_{r 1}$ increases proportionally with the wire current. When $f_{r 1}$ approaches zero, the transverse component of $\vec{B}_{n}$ has no time to rotate or vary within the effective flight path-length for nuclear-resonant rotation; thus, its variance vanishes. Conversely, when $f_{r 1}$ approaches unity, the mean transverse component of $\vec{B}_{n}$ nears zero; thus, the variance peaks towards $\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)^{2}$. In between, the variance is approximated by linear interpolation.

We have $\pi y_{r 1} /\left(\pi y_{r 0}\right)=B_{n} \sin \left\langle\theta_{n}\right\rangle / B_{y}^{\prime}$. At the current of 0.067 A that divides the two regimes of current, $\pi y_{r 1}=\pi y_{r 0}=0.33 \times 10^{-3} \mathrm{~m}$, yielding $\pi y_{r 1} /\left(\pi y_{r 0}\right)=1$. As illustrated in Fig. S5, at the current of $0.2 \mathrm{~A}, \pi y_{r 1}$ equals $0.98 \times 10^{-3} \mathrm{~m}$, yielding $\pi y_{r 1} /\left(\pi y_{r 0}\right)=3.0$. At the maximum current of $0.5 \mathrm{~A}, \pi y_{r 1}$ increases to $2.5 \times 10^{-3} \mathrm{~m}$, yielding $\pi y_{r 1} /\left(\pi y_{r 0}\right)=7.5 ; f_{r 1}$ reaches 0.34 . In comparison, the counterpart $f_{r 0}$ during the flight time over $\pi y_{r 0}$ is only $0.34 \times \pi y_{r 0} /\left(\pi y_{r 1}\right)=0.045 \ll 1$; thus, the transverse field is stable, i.e., its angular variation is much less than $2 \pi$. Therefore, the null-point rotation is due to the static and quasi-static transverse field within $\pi y_{r 0}$, whereas the nuclear-resonant rotation is due to the rotating transverse field within $\pi y_{r 1}$. Disrupted by the null-point rotation in combination with the random $\phi_{n 0}$ in the ensemble of atoms (see Eq. 133), the nuclear-resonant rotation contributes to the fraction of flip through the variance (instead of Eq. 139 in the absence of the null-point rotation). Note that one may consider the null-point rotation as a resonant effect but at a zero Larmor frequency.

Combining terms yields

$$
\begin{equation*}
E_{r 1}=\frac{\pi^{2}\left|\gamma_{e}\right|^{2} \gamma_{n} z_{a}^{3}}{4 v^{3}} \frac{B_{e}\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)^{5}}{B_{y}^{3}} . \tag{147}
\end{equation*}
$$

Expressing $B_{y}^{\prime}$ in terms of the current, $I$, using Eq. 120 and 122 gives

$$
\begin{equation*}
E_{r 1}=\frac{\mu_{0}^{3}\left|\gamma_{e}\right|^{2} \gamma_{n}}{32 \pi v^{3}} \frac{B_{e}\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)^{5}}{\left(B_{r}+B_{n} \cos \left\langle\theta_{n}\right\rangle\right)^{6}} I^{3} . \tag{148}
\end{equation*}
$$

Using the dimensionless adiabaticity parameters $k_{0}$ (Eq. 125) and $k_{1}$ (Eq. 126), we simplify Eq. 141 and 142 to

$$
\begin{equation*}
E_{r 0}=\pi k_{0} \sqrt{1+k_{1} / k_{0}}=\pi \sqrt{k_{0}^{2}+k_{0} k_{1}} \tag{149}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{r 1}=\frac{1}{2}\left[\pi k_{1}\right]^{2}\left\{f_{r 1}\right\} \tag{150}
\end{equation*}
$$

Thus, $E_{r 0}$ is due to the quadrature sum of $k_{0}$ and $\sqrt{k_{0} k_{1}}$, and $E_{r 1}$ is due to $k_{1}$. Eq. 146 can be rewritten as $f_{r 1}=\frac{\pi z_{a}}{v T_{n}} \sqrt{k_{1} / k_{0}}$. Substitution of the above two equations into Eq. 140 with $E_{i}=0$ yields the fraction of flip

$$
\begin{equation*}
W_{4}=\exp \left(-\pi \sqrt{k_{0}^{2}+k_{0} k_{1}}-\frac{1}{2}\left[\pi k_{1}\right]^{2} f_{r 1}\right) \tag{151}
\end{equation*}
$$

which produces Curve 4 in Fig. 4. If $f_{r 1}=0, W_{4}$ reduces to

$$
\begin{equation*}
W_{3}=\exp \left(-\pi \sqrt{k_{0}^{2}+k_{0} k_{1}}\right) \tag{152}
\end{equation*}
$$

which produces Curve 3 in Fig. 4. If $k_{1}=0, W_{3}$ reduces to

$$
\begin{equation*}
W_{2}=\exp \left(-\pi k_{0}\right) \tag{153}
\end{equation*}
$$

which produces Curve 2 in Fig. 4. If $k_{0}$ is computed without including the correction to the remnant field (Eq. 119), $k_{0}$ reduces to $k_{m}$; thus, $W_{2}$ reduces to

$$
\begin{equation*}
W_{1}=\exp \left(-\pi k_{m}\right) \tag{154}
\end{equation*}
$$

which produces Curve 1 in Fig. 4. Taking the square root of $W_{1}$ leads to

$$
\begin{equation*}
W_{m}=\exp \left(-\pi k_{m} / 2\right) \tag{155}
\end{equation*}
$$

which produces Curve $m$ in Fig. 4.
The induction exponent is estimated as

$$
\begin{equation*}
E_{i}=4 k_{i} \int_{-T_{f} / 2}^{+T_{f} / 2}\left|\dot{\phi}_{e}\right| d t \tag{156}
\end{equation*}
$$

where $T_{f}$ denotes the entire flight time corresponding to a path-length of 16.3 mm in the inner rotation chamber. The field generated from the wire contributes to induction. At the nearest point to the wire along the atomic path, the magnetic flux density is

$$
\begin{equation*}
B_{w}(0)=\frac{\mu_{0} I}{2 \pi z_{a}}, \tag{157}
\end{equation*}
$$

which reaches $9.5 \times 10^{-4} \mathrm{~T}$ at the maximum current of 0.5 A and $z_{a}$ of $1.05 \times 10^{-4} \mathrm{~m}$. Therefore, the strongest $B_{w}(0)$ is 80 times $\left(=9.5 \times 10^{-4} / 0.119 \times 10^{-4}\right)$ greater than $B_{n}, 23$ times $(=$ $9.5 \times 10^{-4} / 0.42 \times 10^{-4}$ ) stronger than $B_{r}$, but 59 times $\left(=558 \times 10^{-4} / 9.5 \times 10^{-4}\right.$ ) weaker than $B_{e}$. Also, $B_{w}(0)$ is $\sim 300$ times weaker than a typical main field $[5,37](\geq 0.3 \mathrm{~T})$ in SternGerlach experiments (see Paragraph 2 in Discussion). Without including the induction exponent, CQD predicts the Frisch-Segrè experimental observation well already (Fig. 1 or Fig. 4). Consequently, the induction effect in the inner rotation chamber is initially neglected. However, it is included here for completeness and for the estimation of $k_{i}$. Below, zero time $t$ is set to when an atom reaches the nearest point to the wire.

From Eq. 156, we estimate $E_{i}$ by using the $z$ component of the field along the path:

$$
\begin{equation*}
E_{i}=4 k_{i}\left|\gamma_{e}\right| \int_{-T_{f} / 2}^{+T_{f} / 2}\left|\frac{B_{w}(0) v t / z_{a}}{1+\left(v t / z_{a}\right)^{2}}\right| d t=4 k_{i}\left|\gamma_{e}\right| B_{w}(0) \frac{z_{a}}{v} \ln \left(\frac{T_{f} v}{2 z_{a}}\right) . \tag{158}
\end{equation*}
$$

Substituting Eq. 157 gives

$$
\begin{equation*}
E_{i}=k_{i} \frac{2 \mu_{0}\left|\gamma_{e}\right|}{\pi v} \ln \left(\frac{T_{f} v}{2 z_{a}}\right) I . \tag{159}
\end{equation*}
$$

Expressing in terms of the current, $I$, yields

$$
\begin{equation*}
W_{\mathrm{cqd}}=\exp \left[-\sqrt{\left(c_{r 0} / I\right)^{2}+c_{r s}^{2}}-c_{r 1} I^{3}-c_{r i} I\right] \tag{160}
\end{equation*}
$$

where

$$
\begin{gather*}
c_{r 0}=2 \pi^{2}\left|\gamma_{e}\right|\left(B_{r}+B_{n} \cos \left\langle\theta_{n}\right\rangle\right)^{2} z_{a}^{2} /\left(\mu_{0} v\right),  \tag{161}\\
c_{r s}=\pi\left|\gamma_{e}\right| B_{n} \sin \left\langle\theta_{n}\right\rangle z_{a} / v,  \tag{162}\\
c_{r 1}=\frac{\mu_{0}^{3}\left|\gamma_{e}\right|^{2} \gamma_{n}}{32 \pi v^{3}} B_{e}\left(B_{n} \sin \left\langle\theta_{n}\right\rangle\right)^{5} /\left(B_{r}+B_{n} \cos \left\langle\theta_{n}\right\rangle\right)^{6}, \tag{163}
\end{gather*}
$$

and

$$
\begin{equation*}
c_{r i}=k_{i} \frac{2 \mu_{0}\left|\gamma_{e}\right|}{\pi v} \ln \left(\frac{T_{f} v}{2 z_{a}}\right) . \tag{164}
\end{equation*}
$$

The four coefficients represent null-point rotation, rotation saturation, nuclear-resonant rotation, and induction rotation of the polar angle, respectively. While null-point rotation increases $W_{\text {cqd }}$ with increasing current, nuclear-resonant rotation does the opposite.

If $k_{i}=0$, Eq. 160 reduces to

$$
\begin{equation*}
W_{4}=\exp \left[-\sqrt{\left(c_{r 0} / I\right)^{2}+c_{r s}^{2}}-c_{r 1} I^{3}\right], \tag{165}
\end{equation*}
$$

which is equivalent to Eq. 151 . Further, if $c_{r 1}=0$, we reach

$$
\begin{equation*}
W_{3}=\exp \left[-\sqrt{\left(c_{r 0} / I\right)^{2}+c_{r s}^{2}}\right], \tag{166}
\end{equation*}
$$

which is another form of Eq. 152. Merging the above two equations, we obtain

$$
\begin{equation*}
W_{4}=W_{3} \exp \left(-c_{r 1} I^{3}\right) \tag{167}
\end{equation*}
$$

Finally, if $c_{r s}=0$, we obtain

$$
\begin{equation*}
W_{2}=\exp \left(-c_{r 0} / I\right), \tag{168}
\end{equation*}
$$

which is the same as Eq. 153.

Appendix 6. Uncertainty relation
If the initial co-quanta are isotropically distributed, CQD reproduces the quantum mechanical uncertainty relation.

First, we predict the expectation of the spin-angular-momentum projection, $\left\langle s_{y}\right\rangle$, along $y$, the direction of the atomic beam. The CQD prediction expression for the $y$ axis is (Eq. 13)

$$
\begin{equation*}
\left|\hat{\mu}_{e} \Subset \hat{\mu}_{n}\right\rangle_{y}=C_{+y}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right)|+y\rangle+C_{-y}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right) \exp \left(i \phi_{e y}\right)|-y\rangle, \tag{169}
\end{equation*}
$$

where $\phi_{e y}$ denotes the azimuthal angle of $\vec{\mu}_{e}$ about $y$. Following Appendix 3 yields the wave function,

$$
\begin{equation*}
\left|\hat{\mu}_{e}\right\rangle_{y}=\cos \frac{\theta_{e y}}{2}|+y\rangle+\sin \frac{\theta_{e y}}{2} \exp \left(i \phi_{e y}\right)|-y\rangle \tag{170}
\end{equation*}
$$

where $\theta_{e y}$ denotes the polar angle of $\vec{\mu}_{e}$ relative to $y$. Note that $|+y\rangle$ here denotes "up" for magnetic moment and hence "down" for electron spin, and $|-y\rangle$ denotes the opposite state.

The expectation is

$$
\begin{equation*}
\left\langle s_{y}\right\rangle=-\frac{\hbar}{2} \cos ^{2} \frac{\theta_{e y}}{2}+\frac{\hbar}{2} \sin ^{2} \frac{\theta_{e y}}{2}=-\frac{\hbar}{2} \cos \theta_{e y} . \tag{171}
\end{equation*}
$$

Second, we measure along $z$. The CQD prediction expression for the $z$ axis is (Eq. 13)

$$
\begin{equation*}
\left.\left|\hat{\mu}_{e}\left(\hat{\mu}_{n}\right\rangle_{z}=C_{+z}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right)\right|+z\right\rangle+C_{-z}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right) \exp \left(i \phi_{e z}\right)|-z\rangle \tag{172}
\end{equation*}
$$

We similarly derive the wave function (Appendix 3),

$$
\begin{equation*}
\left|\hat{\mu}_{e}\right\rangle_{z}=\cos \frac{\theta_{e z}}{2}|+z\rangle+\sin \frac{\theta_{e z}}{2} \exp \left(i \phi_{e z}\right)|-z\rangle \tag{173}
\end{equation*}
$$

where $\theta_{e z}$ and $\phi_{e z}$ denote the polar and azimuthal angles in relation to $z$.
We compute the standard deviation, $\Delta s_{z}$, as follows:

$$
\begin{align*}
& \left\langle s_{z}\right\rangle=-\frac{\hbar}{2} \cos ^{2} \frac{\theta_{e z}}{2}+\frac{\hbar}{2} \sin ^{2} \frac{\theta_{e z}}{2}=-\frac{\hbar}{2} \cos \theta_{e z}  \tag{174}\\
& \left\langle s_{z}^{2}\right\rangle=\left(-\frac{\hbar}{2}\right)^{2} \cos ^{2} \frac{\theta_{e z}}{2}+\left(\frac{\hbar}{2}\right)^{2} \sin ^{2} \frac{\theta_{e z}}{2}=\left(\frac{\hbar}{2}\right)^{2} \tag{175}
\end{align*}
$$

and

$$
\begin{equation*}
\Delta s_{z}=\sqrt{\left\langle s_{Z}^{2}\right\rangle-\left\langle s_{Z}\right\rangle^{2}}=\frac{\hbar}{2} \sin \theta_{e z} \tag{176}
\end{equation*}
$$

At this point, $\hat{\mu}_{e}$ has collapsed to $\pm z$; thus, the polar angle relative to $x, \theta_{e x}=\pi / 2$. Now, the coquantum distribution follows the heart shape (Eq. 24).

Third, we measure along $x$. The CQD prediction expression for the $x$ axis is (Eq. 13)

$$
\begin{equation*}
\left|\hat{\mu}_{e} \mathbb{C} \hat{\mu}_{n}\right\rangle_{x}=C_{+x}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right)|+x\rangle+C_{-x}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right) \exp \left(i \phi_{e x}\right)|-x\rangle . \tag{177}
\end{equation*}
$$

Invoking $\theta_{e x}=\pi / 2$, the heart shape (Eq. 24), and the identity $\cos \theta_{n z}=\sin \theta_{n x} \sin \phi_{n x}$, we follow Appendix 3 to similarly obtain for the $\pm z$ branch

$$
\begin{equation*}
\left\langle C_{+x}\right\rangle_{n}^{2}=\int_{\pi / 2}^{\pi} \int_{0}^{2 \pi} \frac{1 \mp \cos \theta_{n z}}{4 \pi} \sin \theta_{n x} d \phi_{n x} d \theta_{n x}=\frac{1}{2} \tag{178}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle C_{-x}\right\rangle_{n}^{2}=\int_{0}^{\pi / 2} \int_{0}^{2 \pi} \frac{1 \mp \cos \theta_{n z}}{4 \pi} \sin \theta_{n x} d \phi_{n x} d \theta_{n x}=\frac{1}{2} \tag{179}
\end{equation*}
$$

The even split between the two $\pm x$ branches is because the heart shape associated with either of the $\pm z$ branches is rotationally symmetric about the $z$ axis. Thus, the wave function is

$$
\begin{equation*}
\left|\hat{\mu}_{e}\right\rangle_{x}=\frac{1}{\sqrt{2}}|+x\rangle+\frac{1}{\sqrt{2}} \exp \left(i \phi_{e x}\right)|-x\rangle \tag{180}
\end{equation*}
$$

We derive the standard deviation, $\Delta s_{x}$, as follows:

$$
\begin{gather*}
\left\langle s_{x}\right\rangle=-\frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}\right)^{2}+\frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}\right)^{2}=0  \tag{181}\\
\left\langle s_{x}^{2}\right\rangle=\left(-\frac{\hbar}{2}\right)^{2}\left(\frac{1}{\sqrt{2}}\right)^{2}+\left(\frac{\hbar}{2}\right)^{2}\left(\frac{1}{\sqrt{2}}\right)^{2}=\left(\frac{\hbar}{2}\right)^{2}, \tag{182}
\end{gather*}
$$

and

$$
\begin{equation*}
\Delta s_{x}=\sqrt{\left\langle s_{x}^{2}\right\rangle-\left\langle s_{x}\right\rangle^{2}}=\frac{\hbar}{2} . \tag{183}
\end{equation*}
$$

Fourth, combining Eq. 176 and 183 reaches

$$
\begin{equation*}
\Delta s_{z} \Delta s_{x}=\left(\frac{\hbar}{2} \sin \theta_{e z}\right) \cdot \frac{\hbar}{2} \tag{184}
\end{equation*}
$$

Substituting the identity, $\cos \theta_{e y}=\sin \theta_{e z} \sin \phi_{e z}$, into Eq. 171 yields

$$
\begin{equation*}
\frac{\hbar}{2}\left|\left\langle s_{y}\right\rangle\right|=\left[\left(\frac{\hbar}{2} \sin \theta_{e z}\right) \cdot \frac{\hbar}{2}\right] \cdot\left|\sin \phi_{e z}\right| . \tag{185}
\end{equation*}
$$

Combining the above two equations yields

$$
\begin{equation*}
\Delta s_{z} \Delta s_{x} \cdot\left|\sin \phi_{e z}\right|=\frac{\hbar}{2}\left|\left\langle s_{y}\right\rangle\right| . \tag{186}
\end{equation*}
$$

This uncertainty equality shows that the magnitude of the uncertainty product, $\Delta s_{z} \Delta s_{x}$, depends on not only $\left\langle s_{y}\right\rangle$ but also the initial phase, $\phi_{e z}$, in relation to the first measurement axis. Therefore, the order of the $z-x$ measurements matters.

Finally, invoking $\left|\sin \phi_{e z}\right| \leq 1$ reproduces exactly the familiar quantum mechanical uncertainty inequality for angular momenta,

$$
\begin{equation*}
\Delta s_{z} \Delta s_{x} \geq \frac{\hbar}{2}\left|\left\langle s_{y}\right\rangle\right| \tag{187}
\end{equation*}
$$

which takes on the equal sign when $\phi_{e z}= \pm \pi / 2$.

## Appendix 7. Entanglement

CQD in its current form construes anticorrelated entanglement as a pair of atoms having both opposing $\hat{\mu}_{e}$ 's and $\hat{\mu}_{n}$ 's (Fig. S6). The two atoms are delivered from the entanglement site to two Stern-Gerlach devices. Once the orientation of the external magnetic flux density $\vec{B}_{0}$ is chosen, the two $\hat{\mu}_{e}$ 's are guaranteed to collapse to opposing states according to the branching condition because $\theta_{e 1}+\theta_{e 2}=\pi$ and $\theta_{n 1}+\theta_{n 2}=\pi$. For example, if atom 1 collapses to $+\widehat{B}_{0}$ because $\theta_{n 1}>\theta_{e 1}$, atom 2 automatically collapses to $-\widehat{B}_{0}$ because $\theta_{n 2}<\theta_{e 2}$ (as derived from $\pi-\theta_{n 1}<$ $\pi-\theta_{e 1}$ ). Therefore, the co-quanta propagate with the principal quanta and determine the measurement outcomes.

The CQD prediction expressions for the two atoms are

$$
\begin{equation*}
\left|\hat{\mu}_{e} \Subset \hat{\mu}_{n}\right\rangle_{1}=C_{+}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right)\left|+\hat{B}_{0}\right\rangle_{1}+C_{-}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right) \exp \left(i \phi_{e 1}\right)\left|-\hat{B}_{0}\right\rangle_{1} \tag{188}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\left|-\hat{\mu}_{e}\left(C-\hat{\mu}_{n}\right\rangle_{2}=C_{-}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right)\right|+\hat{B}_{0}\right\rangle_{2}+C_{+}\left(\hat{\mu}_{e}, \hat{\mu}_{n}\right) \exp \left(i \phi_{e 2}\right)\left|-\hat{B}_{0}\right\rangle_{2} . \tag{189}
\end{equation*}
$$

The numeral subscripts denote the two atoms. The joint pre-collapse state function of the atom pair is written as $\left|\hat{\mu}_{e} \Subset \hat{\mu}_{n}\right\rangle_{1} \otimes \mid-\hat{\mu}_{e}\left(\bigcirc-\hat{\mu}_{n}\right\rangle_{2}$, where $\otimes$ denotes tensor product [59]. Because $C_{+} \cdot$ $C_{-}=0$ and $C_{ \pm} \cdot C_{ \pm}=C_{ \pm}$, the joint CQD prediction expression of the atom pair is given by

$$
\begin{gather*}
\left|\hat{\mu}_{e} \complement \hat{\mu}_{n}\right\rangle_{1} \otimes\left|-\hat{\mu}_{e} \Subset-\hat{\mu}_{n}\right\rangle_{2} \\
=C_{+}\left|+\hat{B}_{0}\right\rangle_{1} \otimes\left|-\hat{B}_{0}\right\rangle_{2}+C_{-} \exp (i \Delta \phi)\left|-\hat{B}_{0}\right\rangle_{1} \otimes\left|+\hat{B}_{0}\right\rangle_{2}, \tag{190}
\end{gather*}
$$

where $\Delta \phi=\phi_{e 1}-\phi_{e 2}$ and the common phase is removed.


Fig. S6. Entanglement of the electron magnetic moments of two atoms. (a) At entanglement site. (b) - (d) At two Stern-Gerlach devices with three orientations of $\vec{B}_{0}$. $e$, electron magnetic moment; $n$, nuclear magnetic moment; subscripts 1 and 2 , atoms 1 and 2 . The short arrows indicate the directions of collapse determined by the branching condition, where the polar angles are relative to $\vec{B}_{0}$. In $d$, to facilitate comparison, one may mirror $\hat{\mu}_{e}$ about $\vec{B}_{0}$ because $\hat{\mu}_{e}$ precesses much faster than $\hat{\mu}_{n}$.

If $\Delta \phi$ is constant for a given experimental configuration, ensemble averaging Eq. 190, denoted by $\left\rangle_{n, e}\right.$, yields the familiar quantum mechanical entangled wave function,

$$
\begin{equation*}
|\psi\rangle=\left\langle C_{+}\right\rangle_{n, e}\left|+\widehat{B}_{0}\right\rangle_{1} \otimes\left|-\widehat{B}_{0}\right\rangle_{2}+\left\langle C_{-}\right\rangle_{n, e} \exp (i \Delta \phi)\left|-\widehat{B}_{0}\right\rangle_{1} \otimes\left|+\widehat{B}_{0}\right\rangle_{2} \tag{191}
\end{equation*}
$$

If both $\hat{\mu}_{n}$ and $\hat{\mu}_{e}$ of individual atoms are isotropically distributed, Appendix 3 provides $\left\langle C_{ \pm}\right\rangle_{n, e}=$ $1 / \sqrt{2}$, yielding

$$
\begin{equation*}
|\psi\rangle=\frac{1}{\sqrt{2}}\left|+\widehat{B}_{0}\right\rangle_{1} \otimes\left|-\widehat{B}_{0}\right\rangle_{2}+\frac{1}{\sqrt{2}} \exp (i \Delta \phi)\left|-\widehat{B}_{0}\right\rangle_{1} \otimes\left|+\widehat{B}_{0}\right\rangle_{2} . \tag{192}
\end{equation*}
$$

Here, the key to producing the above entangled wave function is the mutual exclusivity of the binary coefficients: $C_{+} \cdot C_{-}=0$. The pre-collapse "product state" in CQD, $\left|\hat{\mu}_{e} ® \hat{\mu}_{n}\right\rangle_{1} \otimes$ $\left|-\hat{\mu}_{e} \Subset-\hat{\mu}_{n}\right\rangle_{2}$, averages to an entangled wave function (i.e., not a product state). One can adapt the above derivation for correlated instead of anticorrelated entanglement.

A future direction is to explore CQD in relation to Bell's theorem [60]. An ideal experiment would follow the above derivation, where entangled pairs of alkali-metal atoms are delivered to two Stern-Gerlach devices with independent quantization axes. Shin et al. [61] published in 2019 an experiment in this direction. However, the quantization axes could not be controlled independently, and Bose-Einstein condensate helium-4 atoms in the long-lived metastable state $2^{3} S_{1}$ instead of alkali-metal atoms were used. The same group also published in 2022 an experiment on the Bell inequality for motional degrees of freedom of massive particles, thus far reaching a maximum CHSH-Bell parameter of $S=1.1$ [62].

Appendix 8. Two-stage Stern-Gerlach apparatus with a varying angle between the quantization axes
CQD can potentially be further verified with a two-stage Stern-Gerlach apparatus with a varying angle between the quantization axes. As usual, the first stage polarizes the atoms to the $+z$ state; however, the second stage is rotated by an arbitrary angle $\alpha$ about the $y$ axis (the atomic beam axis). Below, the coordinates of the second stage are denoted with primes.

Using the heart-shaped angular distribution of the co-quanta (Eq. 24) and invoking both $\theta_{e z^{\prime}}=\alpha$ and

$$
\begin{equation*}
\cos \theta_{n z}=\frac{1}{\sqrt{2}}\left(\cos \theta_{n z^{\prime}}-\sin \theta_{n z^{\prime}} \cos \phi_{n z^{\prime}}\right) \tag{193}
\end{equation*}
$$

CQD predicts the probability of collapsing to the $+z^{\prime}$ state as

$$
\begin{equation*}
\left\langle C_{+z^{\prime}}\right\rangle_{n}^{2}=\int_{\alpha}^{\pi} \int_{0}^{2 \pi} \frac{1-\cos \theta_{n z}}{4 \pi} \sin \theta_{n z^{\prime}} d \phi_{n z^{\prime}} d \theta_{n z^{\prime}}=\frac{(1+\cos \alpha)^{2}(2-\cos \alpha)}{4} . \tag{194}
\end{equation*}
$$

In comparison, the standard literature [7,30] predicts $\cos ^{2}(\alpha / 2)$ or $(1+\cos \alpha) / 2$ (Eq. 16). At $\alpha$ of $0, \pi / 2$, and $\pi$, the typical angles in the literature, CQD and the literature predict the same probabilities of $1,1 / 2$, and 0 , respectively. In general, however, the ratio of the two predictions is

$$
\begin{equation*}
R_{+}=\left[9-(2 \cos \alpha-1)^{2}\right] / 8, \tag{195}
\end{equation*}
$$

which holds for $0 \leq \alpha<\pi$ with $\pi$ excluded to avoid $0 / 0$. The ratio peaks at $\alpha$ of $\pi / 3$ with $R_{+}=$ $9 / 8$ then decreases with increasing $\alpha$. At $\alpha=11 \pi / 12$, for example, $R_{+}=0.05$, with CQD predicting a much lower probability.

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